Lecture 15 (10/6/2021)

* Prayer
* Spiritual thought ...

* Method of undetermined coefficients: $y'' + p(t)y' + q(t)y = g(t)$ $(*)$

  Idea : • solve the homogeneous eq. to get the complementary solution y_c

  • find one solution to $(*)$, call it y_p

  based on the form of $g(t)$

Ex:

$$y'' + y' - 2y = e^{-t} \rightsquigarrow \text{test } y_p = Ae^{-t}$$

$$y'' + y' - 2y = t^2 e^t \rightsquigarrow \text{test } y_p = (At^2 + Bt + C)e^{-t}$$

Sometimes this way of guessing may fail :

$$y'' + y' - 2y = e^t \rightsquigarrow \text{test } y_p = Ae^t$$

$$\rightsquigarrow \underbrace{Ae^t + Ae^t - 2Ae^t}_{0} = e^t \quad (!!)$$

Why does it fail? Complementary solution : $y_c = \underbrace{c_1 e^t}_{\text{this was our guess for } y_p} + c_2 e^{-2t}$.

How to fix? $y_p = (At+B)e^t = Ate^t + \underbrace{Be^t}_{\text{again in } y_c}$

$$y_p = Ate^t \text{ : not in } y_c$$

$$y' = Ate^t + Ae^t$$

$$y'' = Ate^t + Ae^t + Ae^t$$

$$y'' + y' - 2y = 3Ae^t \rightsquigarrow A = 1/3$$

Ex : $y'' - 2y' + y = e^t \rightsquigarrow y_c = c_1 e^t + c_2 t e^t$

First guess $y_p = Ae^t$

Second guess $y_p = (At+B)e^t$

Third guess : $y_p = t^2(At+B)e^t$

Ex $y'' + 2y' + 2y = e^{-t}\sin t + e^{t} + \sin t.$

* Complementary sol: $y_c = c_1 e^{-t}\sin t + c_2 e^{t}\cos t$

* Find a particular sol: split the problem into 3 independent problems:

$$y = y_1 + y_2 + y_3$$

$y_1 \longrightarrow y'' + 2y' + 2y = e^{-t}\sin t$

$y_2 \longrightarrow y'' + 2y' + 2y = e^{t} \longrightarrow y_2 = Ae^{t} \rightarrow A = \frac{1}{5} \rightarrow y_2 = \frac{1}{5}e^{t}$

$y_3 \rightarrow y'' + 2y' + 2y = \sin t \longrightarrow y_3 = A\cos t + B\sin t \longrightarrow$ get A, B

$2y' + y = \sin t$

$-2A\sin t + 2B\cos t + A\cos t + B\sin t = \sin t$

$$\begin{cases} -2A + B = 1 \\ 2B + A = 0 \end{cases} \longrightarrow \begin{cases} A = -2/5 \\ B = 1/5 \end{cases}$$

$$y_3 = -\frac{2}{5}\cos t + \frac{1}{5}\sin t$$

$y_1 = e^{-t}(A\cos t + B\sin t) = Ae^{-t}\cos t + Be^{-t}\sin t$

appears in y_c

$y_1 = te^{-t}(A\cos t + B\sin t) = e^{-t}(At\cos t + Bt\sin t)$

$y_1' = e^{-t}(-At\cos t - Bt\sin t + A\cos t - At\sin t + B\sin t + Bt\cos t)$

$= e^{-t}[(B-A)t\cos t - (A+B)t\sin t + A\cos t + B\sin t]$

$y_1'' = e^{-t}[\,\cdots\,]$

$\rightsquigarrow A = -1/2,\ B = 0 \rightsquigarrow y_1 = -\frac{1}{2}te^{-t}\cos t.$

Thus, $y_p = y_1 + y_2 + y_3 = -\frac{1}{2}te^{-t}\cos t + \frac{1}{5}e^{t} - \frac{2}{5}\cos t + \frac{1}{5}\sin t$

Given two solutions to $y'' + p(t)y' + q(t)y = 0.^{(*)}$ How to get a solution to $y'' + p(t)y' + q(t)y = g(t)$?

$y = c_1y_1 + c_2y_2$ solves $(*)$

Idea: allow c_1 and c_2 to depend on t.