Lecture 20 (10/18/2021)

Last time: Linear ODE of n'th order

$$y^{(n)} + p_1(t)\,y^{(n-1)} + p_2(t)\,y^{(n-2)} + \dots + p_n(t)\,y = g(t). \qquad (*)$$

Initial conditions $y(t_0) = y_0,\ y'(t_0) = y_0',\ \dots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$.

The initial value problem has a unique solution.

How to solve $(*)$? We first consider the simple case when $p_1(t), \dots, p_n(t)$ are constants and $g(t) = 0$.

Ex

$$y''' - 2y'' - y' + 2y = 0$$

$$y^{(6)} + y = 0$$

......

Use the same idea as in 2^{nd} order ODE: testing candidate $y = e^{rt}$.

$$\begin{aligned}
2 \times (y &= e^{rt}) \\
(-1) \times (y' &= re^{rt}) \\
(-2) \times (y'' &= r^2e^{rt}) \\
1 \times (y''' &= r^3e^{rt}) \\
\hline
0 &= e^{rt}\underbrace{(r^3 - 2r^2 - r + 2)}_{=0}
\end{aligned}$$

$$(r-1)(r-2)(r+1) = (r^2-1)(r-2) = r^3 - 2r^2 - r + 2$$

$\rightsquigarrow$ get 3 values of r

$r = 1$ is obviously a root, so $r-1$ is a factor.

$$r^3 - 2r^2 - r + 2 = (r-1)(r^2 - r - 2) = (r-1)(r+1)(r-2)$$

$\rightsquigarrow$ $r_1 = -1,\ r_2 = 1,\ r_3 = 2$.

$\rightsquigarrow$ get 3 solutions

$$y_1 = e^{r_1t} = e^{-t}$$

$$y_2 = e^{r_2t} = e^{t}$$

$$y_3 = e^{r_3t} = e^{2t}$$

The three functions y_1, y_2, y_3 are linearly independent. Why?

Two ways to check:

1 $\qquad c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$

$$\rightsquigarrow \quad c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t} = 0 \qquad \forall t$$

Divide both sides by e^{2t}:

$$\underbrace{c_1 e^{-3t}}_{\to 0 \text{ as } t\to\infty} + \underbrace{c_2 e^{-t}}_{\to 0} + \underbrace{c_3}_{=0} = 0$$

$$\rightsquigarrow \quad c_1 e^{-t} + c_2 e^{t} = 0$$

Divide both sides by e^{t}:

$$\underbrace{c_1 e^{-2t}}_{\to 0 \text{ as } t\to\infty} + \underbrace{c_2}_{=0} = 0$$

$$\rightsquigarrow \quad c_1, c_2, c_3 = 0$$

2 Compute the Wronskian: if the Wronskian $\neq 0$ for some t then y_1, y_2, y_3 are linear independent.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{t} & e^{2t} \\ -e^{-t} & e^{t} & 2e^{2t} \\ e^{-t} & e^{t} & 4e^{2t} \end{vmatrix}$$

$$= e^{-t} e^{t} e^{2t} \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{vmatrix}}_{\neq 0} \neq 0$$

All solutions to the ODE are $y = c_1 y_1 + c_2 y_2 + c_3 y_3$

$$= c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}.$$

Ex $y''' - 3y'' + 3y' - y = 0$

Characteristic eq: $r^3 - 3r^2 + 3r - 1 = 0 \rightsquigarrow$ triple root $r = 1$

$\rightsquigarrow e^t$ is a solution. We still need two more.

$$\underbrace{\left(y''' - 3y'' + 3y' - y\right) \times e^{-t}}_{(ye^{-t})''' = 0} = 0$$

$\rightsquigarrow (ye^{-t})'' = c_1$

$\rightsquigarrow (ye^{-t})' = c_1 t + c_2$

$\rightsquigarrow ye^{-t} = c_1 t^2/2 + c_2 t + c_3$

$\rightsquigarrow y = \frac{c_1}{2} \underbrace{t^2 e^t}_{y_3} + c_2 \underbrace{te^t}_{y_2} + c_3 \underbrace{e^t}_{y_1}$

Ex:

Char. eq. $(r^2+2r+2)^2 (r-1)^3 (r+1)$

$r = -1 \pm i$ $\quad$ $r = 1$ $\quad$ $r = -1$

$y_1 = e^{-t} \cos t$ $\quad$ $y_3 = e^t$ $\quad$ $y_4 = e^{-t}$

$y_2 = e^{-t} \sin t$ $\quad$ $y_7 = 6e^t$

$y_5 = t\, e^{-t} \cos t$ $\quad$ $y_8 = t^2 e^t$

$y_6 = t\, e^{-t} \sin t$

General solution: $y = c_1 y_1 + c_2 y_2 + \dots + c_8 y_8$.

* Solve nonhomogeneous ODE with constant coefficients:

$$y''' + 4y' = e^t + \sin t$$

$y = y_c + y_p$

- Find y_c: $\quad y_c''' + 4y_c' = 0$

Char. eq. $\quad r^3 + 4r = 0 \longrightarrow r(r^2+4) = 0$

$\longrightarrow r = 0, \pm 2i$

$y_1 = e^{0t} = 1$

$y_2 = e^{0t}\cos 2t = \cos 2t$

$y_3 = e^{0t}\sin 2t = \sin 2t$

$$y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$$
$$= c_1 + c_2 \cos 2t + c_3 \sin 2t$$

- Find y_p: split the problem into two: $\quad y_p = y_{p1} + y_{p2}$

$y_{p1} \longrightarrow y''' + 4y' = e^t$

$y_{p2} \longrightarrow y''' + 4y' = \sin t$

Guess: $\quad y_{p1} = Ae^t \longrightarrow A = 1/5$

$y_{p2} = B\cos t + C\sin t \longrightarrow B = -1/3, \; C = 0.$

Thus, $\quad y_p = \frac{1}{5}e^t - \frac{1}{3}\cos t$

$$y = y_c + y_p = c_1 + c_2\cos 2t + c_3 \sin 2t + \frac{1}{5}e^t - \frac{1}{3}\cos t.$$