Lecture 21

Tuesday, October 19, 2021 10:02 PM

* Prayer
* Spiritual thought ....

Equations we can solve analytically :

Equations	Tools
$y' = f(x)g(y)$	– Separation of variables
$y' + p(x)y = g(x)$	– Integrating factor
$M + Ny' = g(x)$, where $M_y = N_x$	– exact derivative
$ay'' + by' + cy = g(x)$, where a, b, c are constants	– variation of parameters
$a_1 y^{(n)} + a_2 y^{(n-1)} + \dots + a_n y = g(x)$, where $a_1, \dots, a_n$ are constants	– undetermined coefficients, variation of parameters

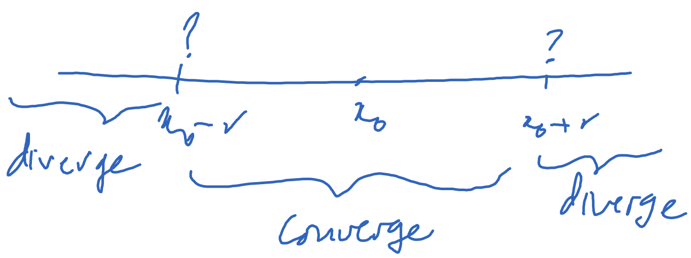
In these methods, a solution y is given in a "closed form". That is, given a value of x, one can obtain a exact value of y by evaluating elementary functions ; polynomial, trigonometric, exponential, logarithm, power functions, and their combinations.

Sometimes, we can't solve for y exactly. For practical purposes, we only need to solve for y approximately.

<u>Power series</u>

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

This series defines a function on the interval $(x_0 - r, x_0 + r)$ where $r \geq 0$ is called the radius of convergence of the power series.



<u>Ex</u>

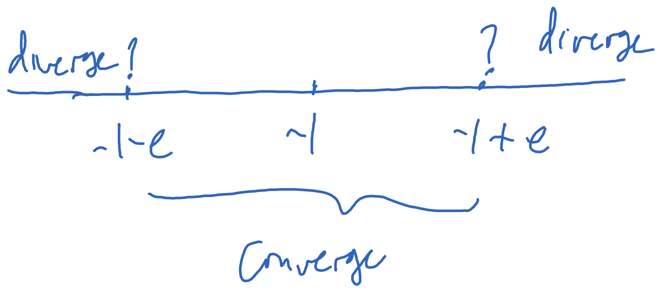
$$\underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{(x+1)^n}{n^n} n!}_{b_n}$$

Note: $$\frac{b_{n+1}}{b_n} = -(x+1)(n+1)\frac{n^n}{(n+1)^{n+1}} = -(x+1)\left(\frac{n}{n+1}\right)^n$$

Thus, $$\left|\frac{b_{n+1}}{b_n}\right| = |x+1| \underbrace{\left(1 - \frac{1}{n+1}\right)^n}_{\to 1/e} \longrightarrow \frac{|x+1|}{e} \text{ as } n \to \infty.$$

If $\dfrac{|x+1|}{e} > 1$ then $\sum b_n$ diverges.

If $\dfrac{|x+1|}{e} < 1$ then $\sum b_n$ converges.



Application:

$$y'' + xy = e^x, \quad y(0) = 0, \; y'(0) = 1.$$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y(0) = 0 \rightsquigarrow a_0 = 0.$$

$$y'(0) = 1 \rightsquigarrow a_1 = 1.$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$y'' + xy = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_{n-1} \right] x^n = e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\rightsquigarrow a_2 = 1, \quad (n+2)(n+1) a_{n+2} + a_{n-1} = \frac{1}{n!}$$