Lecture 22

Friday, October 22, 2021 1:17 AM

* Prayer

* Spiritual thought....

* Application of power series:

Ex $y'' + xy = e^x, \quad y(0) = 0, \; y'(0) = 1.$

$\underbrace{y = \sum_{n=0}^{\infty} a_n x^n}_{\text{Taylor series of } y}$ $\qquad y(0) = 0 \rightsquigarrow a_0 = 0.$

$y'(0) = 1 \rightsquigarrow a_1 = 1.$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$y'' + xy = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_{n-1} \right] x^n = e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$\rightsquigarrow 2a_2 = 1, \quad (n+2)(n+1) a_{n+2} + a_{n-1} = \frac{1}{n!}$

What are a_3 and a_4?

Use Mathematica to plot $P_0(x) = a_0$

$P_1(x) = a_0 + a_1 x$

$P_2(x) = a_0 + a_1 x + a_2 x^2$

$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$P_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$

$\underline{\underline{\text{Ex}}}$ Solve the ODE $y'' + xy = e^x$ (without the initial conditions).

Write $y = \sum_{n=0}^{\infty} a_n x^n$.

Still get

$$a_{n+2} = \frac{\frac{1}{n!} - a_{n-1}}{(n+1)(n+2)}.$$

We need a_0 and a_1. Note: a_2 is determined from the ODE:

$$\underbrace{y''(0)}_{2a_2} + \underbrace{0\,y(0)}_{0} = \underbrace{e^0}_{1} \longrightarrow a_2 = \frac{1}{2}$$

Set $a_0 = 0$, $a_1 = 1 \rightsquigarrow$ get y_1

Set $a_0 = 1$, $a_1 = 0 \rightsquigarrow$ get y_2

Note: $W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$

Thus, y_1 and y_2 form a fundamental set of solutions.

Second method to deal with Example 1:

$$y'' + xy = e^x, \quad y(0) = 0, \quad y'(0) = 1.$$

Write $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n$.

If we want to find only the first few coefficients a_n's, then we don't need to equalize coefficients. Instead, we observe that

$$a_n = \frac{y^{(n)}(0)}{n!}.$$

Thus $\displaystyle a_2 = \frac{y''(0)}{2} = \frac{1}{2}$ (from the ODE).

$$a_3 = \frac{y'''(0)}{6}$$

$$\frac{d}{dx}\left(y'' + xy = e^x\right) \rightsquigarrow y''' + xy' + y = e^x \quad (*)$$

Plug $x=0$: $\quad y'''(0) + 0 + \underbrace{y(0)}_{0} = \underbrace{e^0}_{1} \rightsquigarrow y'''(0) = 1 \rightsquigarrow a_3 = \frac{1}{6}$

$$a_4 = \frac{y^{(4)}(0)}{24}.$$

Differentiating $(*)$:

$$y^{(4)} + xy'' + y' + y' = e^x$$

Plug $x=0$:

$$24a_4 + 0 + \underbrace{2y'(0)}_{2} = \underbrace{e^0}_{1} \rightarrow a_4 = -\frac{1}{24}.$$