Lecture 25 (10/29/2021)

* Prayer
* Spiritual thought ....
* New tool to solve an ODE: using integral transform

  Differential equation $L(y) = g$

  Transform:
  $$Y(s) = \int y(x) K(x,s)\, dx$$

  Equation of y turns into an equation of Y.

Transform	$K(x,s)$
Laplace	e^{-ts}
Fourier	e^{-its}
Z	s^{-x}
Mellin	x^{s-1}
Hilbert	$\frac{1}{x-s}$
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Laplace transform turns an ODE into an algebraic equation.
It has the advantage of being {simple, algorithmic} $\rightsquigarrow$ suitable for engineering.

Def:
$$\mathcal{L}\{f\}(s) = \underbrace{\int_0^\infty f(x) e^{-xs}\, dx}_{\text{improper integral}} \qquad (s>0)$$

Ex:
$$f(x) = 1$$
$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-xs}\, dx = -\frac{1}{s} e^{-xs} \Big|_{x=0}^{x\to\infty} = -\frac{1}{s}0 - \left(-\frac{1}{s}e^0\right) = \frac{1}{s}$$

Ex $f(x) = \begin{cases} 0, & x < 1 \\ x, & x \geq 1 \end{cases}$

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(x) e^{-xs} dx = \int_1^\infty \underbrace{x}_{u} \underbrace{e^{-xs} dx}_{dv} = x\left(-\frac{1}{s} e^{-xs}\right)\Big|_{x=1}^{x\to\infty} - \int_1^\infty -\frac{1}{s} e^{-xs} dx$$

$$= \frac{1}{s} e^{-s} + \int_1^\infty \frac{1}{s} e^{-xs} dx$$

$$= \frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-xs}\Big|_{x=1}^{x\to\infty}$$

$$= \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-s} = \frac{s+1}{s^2} e^{-s}$$

On Mathematica:

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f[x_] := Piecewise[{ {0, x<1}, {x, x>=1} }]
LaplaceTransform[f[x], x, s]
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* Examples on improper integrals:

1) $\int_0^1 \frac{1}{x^p} dx$ converges if $p < 1$, diverges if $p \geq 1$.

2) $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$, diverges if $p \leq 1$.

3) $\int_0^\infty \frac{1}{\sqrt{x}} e^{-x} dx$ converges

4) $\int_0^\infty \frac{\sin x}{x} e^{-x} dx$ converges

5) $\int_0^\infty \frac{e^{-x}}{x^2} dx$ diverges