Lecture 26 (11/01/2021)

* Prayer
* Spiritual thought

Recall Laplace transform $\mathcal{L}\{f\}(s) = \underbrace{\int_0^\infty f(t)e^{-ts}dt}_{\text{improper integral}}$.

Use Laplace transform to solve an ODE:

$$y' + 3y = e^t$$

Take the Laplace transform of both sides:

$$\mathcal{L}\{y'+3y\} = \mathcal{L}\{e^t\}$$

$$\rightsquigarrow \underbrace{\mathcal{L}\{y'\}}_{sY-y(0)} + 3\underbrace{\mathcal{L}\{y\}}_{Y} = \frac{1}{s-1}$$

$$\rightsquigarrow Y(s) = \frac{\frac{1}{s-1}+y(0)}{s+3} \qquad (*)$$

A nice feature of Laplace transform is that one can relate $\mathcal{L}\{y^{(n)}\}$ with $\mathcal{L}\{y\}$ in a simple way.

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$ (do integration-by-part once)

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$ (do integration-by-part twice)

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$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - s\,y^{(n-2)}(0) - y^{(n-1)}(0).$$

This formula will be provided on the exam.

Ex: $y' + 3y = e^t$, $y(0) = 1$

$$(*) \rightsquigarrow Y(s) = \frac{\frac{1}{s-1}+1}{s+3} = \frac{s}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$ (fractional decomposition)

Get $A = 1/4$, $B = 3/4$.

$$\mathcal{L}\{y\} = \frac{1/4}{s-1} + \frac{3/4}{s+3}$$

Look up the table:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Thus, $\mathcal{L}\{y\} = \frac{1}{4}\mathcal{L}\{e^{t}\} + \frac{3}{4}\mathcal{L}\{e^{-3t}\} = \mathcal{L}\{\frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}\}$.

Therefore, $y = \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$.

The nice thing about Laplace transform method is that the initial conditions are already incorporated in the Laplace transform.

Ex $y''' + 5y'' + 12y' + 8y = e^{-t}$, $y(0) = y'(0) = y''(0) = 0$.

Take Laplace transform of both sides:

$$\mathcal{L}\{y^{(3)}\} = s^3 Y - s^2 0 - s0 - 0 = s^3 Y$$

$$(\mathcal{L}\{y''\} = s^2 Y) \times 5$$

$$(\mathcal{L}\{y'\} = sY) \times 12$$

$$(\mathcal{L}\{y\} = Y) \times 8$$

$$\mathcal{L}\{LHS\} = (s^3 + 5s^2 + 12s + 8)Y = (s+1)(s^2+4s+8)Y$$

$$\mathcal{L}\{RHS\} = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Thus,

$$Y = \frac{1}{(s+1)^2(s^2+4s+8)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4s+8}$$

Get B by multiplying both sides by $(s+1)^2$ and then plug $s=-1$: $B = \frac{1}{5}$

Find A, B, C: plug in 3 values of s to get 3 eqs to solve for A, C, D

On Mathematica: use the command Apart. $\rightsquigarrow$ $A = -2/25$, $C = 2/25$, $D = 1/25$

$$Y = \mathcal{L}\{y\} = -\frac{2}{25}\frac{1}{s+1} + \frac{1}{5}\frac{1}{(s+1)^2} + \frac{1}{25}\frac{2s+1}{s^2+4s+8}$$

$$= -\frac{2}{25}\mathcal{L}\{e^{-t}\} + \frac{1}{5}\mathcal{L}\{te^{-t}\} + \frac{1}{25}\underbrace{\frac{2s+1}{(s+2)^2+2^2}}_{(*)}$$

Note :

$$(*) = \frac{2(s+2)-3}{(s+2)^2+2^2} = 2\underbrace{\frac{s+2}{(s+2)^2+2^2}}_{\mathcal{L}\{e^{-2t}\cos 2t\}} - 3\underbrace{\frac{1}{(s+2)^2+2^2}}_{\frac{1}{2}\mathcal{L}\{e^{-2t}\sin 2t\}}$$

Therefore,

$$\mathcal{L}\{y\} = \mathcal{L}\left\{-\frac{2}{25}e^{-t} + \frac{1}{5}te^{-t} + \frac{2}{25}e^{-2t}\cos 2t - \frac{3}{50}e^{-2t}\sin 2t\right\}.$$

$$y = -\frac{2}{25}e^{-t} + \frac{1}{5}te^{-t} + \frac{2}{25}e^{-2t}\cos 2t - \frac{3}{50}e^{-2t}\sin 2t$$