Lecture 28 (11/05/2021)

* Prayer
* Spiritual thought

* Laplace transform of piecewise functions:

$$\boxed{\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f\}} \qquad \mathcal{L}^{-1}\{e^{-cs}F\} = u_c(t)f(t-c)$$

Ex: $g(t) = u_1(t)\,t^2 + u_2(t)\,e^t = \begin{cases} 0 & \text{if } t<1 \\ t^2 & \text{if } 1<t<2 \\ t^2+e^t & \text{if } t>2 \end{cases}$

$$\mathcal{L}\{g\} = \mathcal{L}\{u_1(t)t^2\} + \mathcal{L}\{u_2(t)e^t\}$$
$$= \mathcal{L}\{u_1(t)f_1(t-1)\} + \mathcal{L}\{u_2(t)f_2(t-2)\}$$

$f_1(t-1) = t^2 \rightsquigarrow f_1(t) = (t+1)^2$

$f_2(t-2) = e^t \rightsquigarrow f_2(t) = e^{t+2}$

$$\mathcal{L}\{g\} = e^{-s}\mathcal{L}\{f_1\} + e^{-2s}\mathcal{L}\{f_2\} = e^{-s}\mathcal{L}\{t^2+2t+1\} + e^{-2s}e^2\mathcal{L}\{e^t\}$$
$$= e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right) + e^{2-2s}\frac{1}{s-1}$$
$$= \frac{e^{-s}(2+s+s^2)}{s^3} + \frac{e^{2-2s}}{s-1}$$

Ex Solve the initial value problem

$$\begin{cases} y'' + 2y' + 2y = f(t) \\ y(0) = y'(0) = 0 \end{cases}$$

$$f(t) = \begin{cases} \cos t & \text{if } t<\pi \\ 0 & \text{if } t>\pi \end{cases}$$

Apply Laplace's transform on both sides:

$$\underbrace{\mathcal{L}\{y''\}}_{s^2 Y} + 2\underbrace{\mathcal{L}\{y'\}}_{sY} + 2\underbrace{\mathcal{L}\{y\}}_{Y} = \mathcal{L}\{f\}$$

$$\mathcal{L}\{f\} = \mathcal{L}\{\cos t - u_{\pi}(t)\cos t\} = \frac{s}{s^2+1} - \mathcal{L}\{u_{\pi}(t)\cos t\}$$

$$\cos t = f_1(t-\pi) \rightsquigarrow f_1(t) = \cos(t+\pi) = -\cos t$$

Thus,

$$\mathcal{L}\{f\} = \frac{s}{s^2+1} - e^{-\pi s}\mathcal{L}\{f_1\} = \frac{s}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$$

Then $$Y = \frac{s}{(s^2+1)(s^2+2s+2)} + e^{-\pi s}\frac{s}{(s^2+1)(s^2+2s+2)}$$

Partial fractional decomposition:

$$\underbrace{\frac{s}{(s^2+1)(s^2+2s+2)}}_{Y_1(s)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2} \qquad \left(A = \frac{1}{5}, B = \frac{2}{5}, C = -\frac{1}{5}, D = \frac{-4}{5}\right)$$

$$\mathcal{L}^{-1}\{Y_1\} = A\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + C\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + (D-C)\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t = g(t)$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{Y_1\} + \mathcal{L}^{-1}\{e^{-\pi s}Y_1\} = g(t) + u_{\pi}(t)\,g(t-\pi)$$