Lecture 29 (11/10/2021)

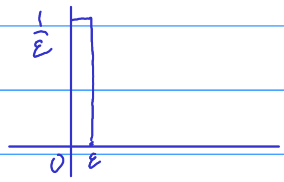
* Prayer
* Spiritual thought

* Impulse forcing:

$$y'' + 2y' + 5y = \underbrace{F(t)}_{\text{external force}}$$

Let us describe the external forcing as follows: we push our finger down at time 0 with the $\underbrace{\text{"strength"}}_{\text{impulse}}$ 1: $\int_0^{\varepsilon} F(t)\,dt = 1$

Then release (no force): $F(t) = 0$ if $t > \varepsilon$.



If ε is very small, then F has to be very large on the interval $(0, \varepsilon)$.

To idealize F, we must allow f to have value ∞.

$$\int_{-\infty}^{\infty} F(t)\,dt = 1, \quad F(t) = \begin{cases} \infty & \text{when } t = 0 \\ 0 & \text{when } t \neq 0 \end{cases}$$

This is not a normal function. It is known as a measure or distribution.

Notation: $F(t) = \delta(t)$: unit impulse function.

$$\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t) e^{-st}\,dt = \int_0^{\varepsilon} \delta(t) \underbrace{e^{-st}}_{\approx 1}\,dt \approx \int_0^{\varepsilon} \delta(t)\,dt = 1$$

Unit impulse function at t_0:

$\delta(t - t_0)$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} \delta(t-t_0) e^{-ts}\,dt = \int_{t_0}^{t_0+\varepsilon} \delta(t-t_0) \underbrace{e^{-ts}}_{\approx e^{-t_0 s}}\,dt = e^{-t_0 s} \underbrace{\int_{t_0}^{t_0+\varepsilon} \delta(t-t_0)\,dt}_{=1}$$

$$= e^{-t_0 s}$$

Ex: $y'' + 2y' + 5y = \delta(t-1)$, $y(0) = y'(0) = 0$

Take Laplace transform: $s^2 Y + 2sY + 5Y = e^{-s} \rightsquigarrow Y = \dfrac{e^{-s}}{s^2 + 2s + 5}$.

$$\leadsto \quad Y = e^{-s} \frac{1}{(s+1)^2 + 2^2} = e^{-s} \mathcal{L}\{f\}$$

$$f(t) = e^{-t} \sin 2t \quad \text{(from the table)}$$

Use the formula $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f\}$ where $c = 1$.

$$Y = e^{-s} \mathcal{L}\{f\} = \mathcal{L}\{u_1(t) f(t-1)\}$$

Therefore,

$$y = \mathcal{L}^{-1}\{Y\} = u_1(t) f(t-1) = u_1(t) e^{-t} \sin 2t.$$

Use Mathematica for illustration.