Lecture 32 (11/17/2021)

* Prayer

* Spiritual thought

Recall an example last time :

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 2y_1 + 3y_2 \end{cases} \rightsquigarrow \text{turn this into a single ODE of } 2^{nd} \text{ order}$$

What if we have a system of 3 equations (3 fluid containers connected to each other) ?

$$\begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = y_1 + 2y_2 - y_3 \\ y_3' = 2y_1 - y_2 + 2y_3 \end{cases} \rightsquigarrow \text{a single ODE of } 3^{rd} \text{ order}$$

(but this task is more difficult)

We will need some background about matrices.

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}}_{Y'} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{Y}$$

*Multiplication of matrices :

$$\underset{m\times n}{A}\ \underset{n\times p}{B}$$

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 7 \\ 1 & 7 & -1 \\ 5 & 0 & 11 \end{bmatrix}$$

* Inverse of a matrix :

$A \to A^{-1}$ A has to be a square matrix

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \qquad A^{-1} = ?$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -4 & -1 & 1 & 0 \\ 0 & -5 & -4 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2/(-5) \\ R_1 \to R_1 - 2R_2}]{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 7/5 & 1/5 & 0 & 2/5 \\ 0 & 1 & 4/5 & 2/5 & 0 & -1/5 \\ 0 & 0 & -4 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3/(-4) \\ R_1 \to R_1 - \frac{7}{5} R_3}]{R_2 \to R_2 + R_3/5} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -3/20 & 7/20 & 2/5 \\ 1/5 & 1/5 & -1/5 \\ 1/4 & -1/4 & 0 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

*Solve a system of linear equations:

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 1 \\ x_1 - x_2 + 2x_3 = 0 \\ -x_1 + 3x_2 - 2x_3 = 2 \end{cases} \rightsquigarrow \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 3 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{b}$$

$$\rightsquigarrow X = A^{-1} b$$

* Differentiate/integrate a matrix:

$$A(t) = \begin{bmatrix} t & t^2 \\ e^t & \sin t \end{bmatrix}, \quad A'(t) = \begin{bmatrix} 1 & 2t \\ e^t & \cos t \end{bmatrix}, \quad \int A(t)\,dt = \begin{bmatrix} t^2/2 + C_1 & t^3/3 + C_2 \\ e^t + C_3 & -\cos t + C_4 \end{bmatrix}$$

$$= \begin{bmatrix} t^2/2 & t^3/3 \\ e^t & -\cos t \end{bmatrix} + \underset{\text{const. matrix}}{C}$$