Lecture 34

Saturday, November 20, 2021 11:33 PM

* Prayer
* Spiritual thought

System of linear ODE can be written in matrix form

$$Y' = P(t)Y + G(t)$$

If $G(t) \equiv 0$ then we have a homogeneous system.

The system $Y' = P(t)Y$ has n linearly independent solutions $Y^{(1)}, Y^{(2)}, \dots, Y^{(n)}$. Here n is the size of the square matrix $P(t)$.

$Y^{(1)}, \dots, Y^{(n)}$ form a fundamental set of solutions. Any solution to $Y' = P(t)Y$ has to be a linear combination of $Y^{(1)}, Y^{(2)}, \dots, Y^{(n)}$.

How to check if $Y^{(1)}, Y^{(2)}, \dots, Y^{(n)}$ are linearly independent?

One way is to compute the Wronskian.

$$W[Y^{(1)} \dots Y^{(n)}] = \det \begin{pmatrix} | & | & & | \\ Y^{(1)}(t) & Y^{(2)}(t) & \cdots & Y^{(n)}(t) \\ | & | & & | \end{pmatrix}$$

If $W = 0$ for some t then they are linear dependent. $(*)$

If $W \neq 0$ for some t then they are linearly independent. $(**)$

Note that (**) is always true for any $Y^{(1)}, \dots, Y^{(n)}$. But (*) is only true when $Y^{(1)}, \dots, Y^{(n)}$ solve the same ODE of the form $Y' = P(t)Y$.

Ex

$$f(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad g(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$W[f,g] = \begin{vmatrix} t & 1 \\ t^2 & t \end{vmatrix} = t^2 - t^2 = 0$$

But $f(t)$ and $g(t)$ are linearly independent.

Ex:

$$f_1 = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}, \quad f_2 = \begin{bmatrix} t \\ 1 \\ t^2 \end{bmatrix}, \quad f_3 = \begin{bmatrix} t \\ t^2 \\ 1 \end{bmatrix}$$

$$W = \begin{vmatrix} 1 & t & t \\ t & 1 & t^2 \\ t^2 & t^2 & 1 \end{vmatrix} \begin{matrix} 1 & t \\ t & 1 \\ t^2 & t^2 \end{matrix} = 1 + t^5 + t^4 - t^3 - t^4 - t^2 = \underbrace{1 + t^5 - t^3 - t^2}_{\neq 0 \text{ when } t = 2.}$$

Thus, f_1, f_2, f_3 are linearly independent.

* Solving a homogeneous system of linear ODEs with constant coefficients:

$$Y' = AY \qquad \text{assuming that } A \text{ is a diagonalizable matrix}$$

$\uparrow$ const. matrix

Ex:

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 2y_1 + 3y_2 \end{cases}$$

Suppose v is an eigenvalue of A and λ is an eigenvector:

test candidate $Y = e^{\lambda t} v$.

$$\left.\begin{aligned} Y' &= \lambda e^{\lambda t} v \\ AY &= A(e^{\lambda t} v) = e^{\lambda t} A v = e^{\lambda t} \lambda v \end{aligned}\right\} \to Y' = AY.$$

<u>Procedure:</u> solve $Y' = AY$ (A diagonalizable)

- Find all the eigenvalues $\lambda_1, \dots, \lambda_n$ of A and corresponding eigenvectors $v_1, \dots, v_n$.
- $Y^{(k)} = e^{\lambda_k t} v_k$ is a solution
- $Y = c_1 Y^{(1)} + \dots + c_n Y^{(n)}$ is the general sol.

<u>Ex</u>

$$\begin{cases} y_1' = 3y_1 - y_2 + 2y_3 \\ y_2' = 3y_1 - y_2 + 6y_3 \\ y_3' = -2y_1 + 2y_2 - 2y_3 \end{cases}$$

$$Y' = AY = \underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}}_{A} Y$$

A has eigenvalues $2, 2, -4$.

Eigenvectors:

$$A - 2I_3 = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$A + 4I_3 = \begin{bmatrix} 7 & -1 & 2 \\ 3 & 3 & 6 \\ -2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to v_3 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

Conclusion:

$$Y = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-4t}\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$