Lecture 35

Monday, November 22, 2021 8:39 PM

* Prayer

* Spiritual thought

* Solving system of ODEs of the form $Y' = AY$, $Y(t_0) = Y_0$

const. matrix, diagonalizable

Ex

$$\begin{cases} y_1' = 3y_1 - y_2 + 2y_3 \\ y_2' = 3y_1 - y_2 + 6y_3 \\ y_3' = -2y_1 + 2y_2 - 2y_3 \end{cases}$$

$y_1(0) = 1$, $y_2(0) = 2$, $y_3(0) = 3$

$$Y' = AY = \underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}}_{A} Y$$

A has eigenvalues 2, 2, −4.

Eigenvectors:

$$A - 2I_3 = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$A + 4I_3 = \begin{bmatrix} 7 & -1 & 2 \\ 3 & 3 & 6 \\ -2 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow v_3 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

Thus,

$$Y = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-4t} \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

Plug $t=0$:

$$Y(0) = c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}-2\\0\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\-3\\2\end{bmatrix}$$

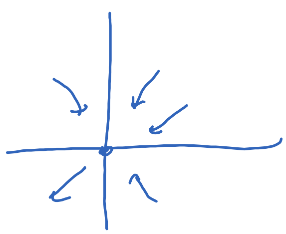
$$\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & -2 & -1\\1 & 0 & -3\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

$$\leadsto \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & -2 & -1\\1 & 0 & -3\\0 & 1 & 2\end{bmatrix}^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}9/2\\4/3\\5/6\end{bmatrix}$$

Autonomous ODE: $y' = f(y) \leadsto$ equilibrium, phaseline

Autonomous system of ODE: $Y' = f(Y) \leadsto$ equilibrium, phase plane

Ex: $\begin{cases} y_1' = y_1 + y_2^2 \\ y_2' = y_1y_2 + 2y_2 \end{cases}$ $\quad\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium.



what kind of equilibrium is this?

Linearize the function f.

$$Y' = \underbrace{J_f(0,0)}_{\text{Jacobian matrix}}\,Y = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}Y$$

Consider this simple system $Y' = AY$. Obviously, $Y = 0$ is an equilibrium.

A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

This ODE has a fundamental set of sol. $Y^{(1)} = v_1 e^{\lambda_1 t}, \dots, Y^{(n)} = v_n e^{\lambda_n t}$.

0 is a hyperbolic equilibrium

If $\lambda_1, \dots, \lambda_n < 0$ then $\vec{0}$ is a stable equilibrium.

If $\lambda_1, \dots, \lambda_n > 0$ then $\vec{0}$ is an unstable equilibrium.

$\vec{0}$ is a node.

If some of λ_k's are positive, some are negatively, then 0 is semistable equilibrium $\longrightarrow$ $\vec{0}$ is a saddle point.

If one of the λ_k's is equal to 0 then $\vec{0}$ is called a non-hyperbolic equilibrium.

Ex: $Y' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} Y \rightsquigarrow \lambda_1 = -1, \quad \lambda_2 = 4$

$Y' = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} Y \rightsquigarrow \lambda_1 = 1, \quad \lambda_2 = 2$

$Y' = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} \rightsquigarrow \lambda_1 = -1, \quad \lambda_2 = -2$

Demonstrate on Mathematica!

```
A = {{-1, 0}, {2, -2}}
x = {x1, x2}
VectorPlot[A.x, {x1, -1, 1}, {x2, -1, 1}],   StreamPlot[....].
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