Lecture 36

Saturday, November 27, 2021 11:03 PM

* Prayer

* Spiritual thought

* Phase portrait: $Y' = f(Y) \rightsquigarrow$ autonomous system of ODEs.

  Ex

$$\begin{cases} y_1' = y_1 + y_2^2 \\ y_2' = y_1 y_2 + 2y_2 \end{cases}$$

Equilibrium states are the zeros of f.

$$f(Y) = 0 \iff \begin{cases} y_1 + y_2^2 = 0 \\ y_1 y_2 + 2y_2 = 0 \end{cases} \rightsquigarrow \begin{cases} y_1 = -2 \\ y_2 = \pm\sqrt{2} \end{cases} \text{ or } \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases}$$

The phase plane of $Y' = f(Y)$ near $(0,0)$ is similar to the phase plane of $Y' = AY$ near $(0,0)$, where $A = J_f(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

* Demonstrate on Mathematica:

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f[y1_, y2_] = {y1 + y2^2, y1 y2 + 2 y2}

A = {{1,0}, {0,2}}

StreamPlot[f[y1, y2], {y1, -1, 1}, {y2, -1, 1}]

StreamPlot[A.{y1, y2}, {y1, -1, 1}, {y2, -1, 1}]
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That's why we only need to focus on the phase plane of $Y' = AY$ where A is a constant matrix.

Recall the procedure to solve $Y' = AY$ (provided that A is a diagonalizable matrix):

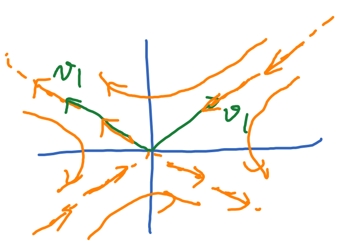
* Find the eigenvalues of A: $\lambda_1, \lambda_2, \dots, \lambda_n$
* Find the corresponding eigenvectors $v_1, v_2, \dots, v_n$.
* Get a fundamental set of solutions $Y^{(1)} = v_1 e^{\lambda_1 t}, \dots, Y^{(n)} = v_n e^{\lambda_n t}$.
* General solution: $Y = c_1 Y^{(1)} + \dots + c_n Y^{(n)}$.

Let's consider the simple case when $n = 2$:

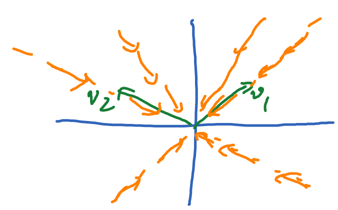
$$Y = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$\left.\begin{array}{l}\text{If } \lambda_1, \lambda_2 > 0: \ \vec{0} = (0,0) \text{ is an unstable equilibrium.} \\ \text{If } \lambda_1, \lambda_2 < 0: \quad \text{" \quad " a stable \quad "}\end{array}\right\} \text{node}$$

If $\lambda_1 < 0, \lambda_2 > 0$ or $\lambda_1 > 0, \lambda_2 < 0$: $\vec{0}$ is a saddle point (unstable).



$\lambda_1 < 0, \lambda_2 > 0$



$\lambda_1, \lambda_2 < 0$

What if λ_1, λ_2 are complex? What would the phase plane look like?

Ex $$Y' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Y$$ $\longrightarrow \vec{0}$ is a center point

Use Mathematica to draw....

Ex $$Y' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} Y$$ $\longrightarrow \vec{0}$ is a spiral point

Use Mathematica to draw....

$\lambda = \alpha + i\beta$

If $\alpha > 0$: unstable spiral point

If $\alpha = 0$: center point

If $\alpha < 0$: stable spiral point

How to solve $Y' = AY$ when A has complex eigenvalues?

- Get $\lambda_1, \lambda_2, v_1, v_2$
- $Y^{(1)} = v_1 e^{\lambda_1 t} = u + iv$
- Fundamental set of sols : $Z^{(1)} = u$, $Z^{(2)} = v$.
- General sol: $Y = c_1 u + c_2 v$.