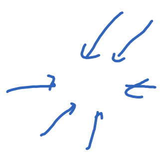
Lecture 37

Wednesday, December 1, 2021 12:03 AM

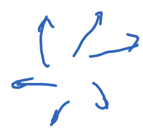
* Prayer
* Spiritual thought

Types of equilibrium states : $Y' = AY$



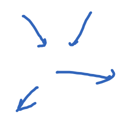
stable node

all eigenvalues < 0



unstable node

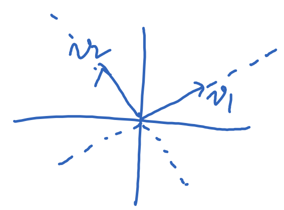
all eigenvalues > 0



saddle point

some eigenvalues < 0

some eigenvalues > 0



What if eigenvalues are complex?

Ex
$$Y' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Y \quad \longrightarrow \vec{0} \text{ is a center point}$$

Use Mathematica to draw....

Ex
$$Y' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} Y \quad \longrightarrow \vec{0} \text{ is a spiral point}$$

Use Mathematica to draw....

$\lambda = \alpha + i\beta$

If $\alpha > 0$: unstable spiral point

If $\alpha = 0$: center point

If $\alpha < 0$: stable spiral point

How to get real solutions (instead of complex solution)?

$$\lambda_1 \xleftrightarrow[\text{conjugate}]{\text{complex}} \lambda_2$$

$$v_1 \xleftrightarrow[\text{Conjugate}]{\text{complex}} v_2$$

$$Y^{(1)} = v_1 e^{\lambda_1 t} \qquad Y^{(2)} = v_2 e^{\lambda_2 t}$$

We replace these two complex solutions by a pair of real solutions.

How? $\quad Y^{(1)} = v_1 e^{\lambda_1 t} = \underbrace{u + iv}_{\text{vectors}}$

$$Y^{(2)} = u - iv$$

$\rightsquigarrow u = \dfrac{Y^{(1)} + Y^{(2)}}{2}$ is a solution

$\rightsquigarrow v = \dfrac{Y^{(1)} - Y^{(2)}}{2i}$ is a solution

} the pair $(Y^{(1)}, Y^{(2)})$ is replaced by the pair (u, v).

Ex:
$$Y' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{A} Y \rightsquigarrow \underbrace{\lambda_1 = 1}_{v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \quad \lambda_{2,3} = 1 \pm i \qquad (\lambda_2 = 1 - i)$$

Find v_2:
$$A - \lambda_2 I_3 = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0, \quad i x_2 + x_3 = 0$$

$$x_2 = 1, \quad x_3 = -i$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$$Y^{(2)} = v_2 e^{\lambda_2 t} = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} e^{(1-i)t} = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} e^t (\cos t - i \sin t)$$

$$= e^t \begin{bmatrix} 0 \\ \cos t - i \sin t \\ i \cos t + \sin t \end{bmatrix}$$

$$= \underbrace{e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}}_{u} + i \underbrace{e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix}}_{v}$$

New fundamental set of sols: $e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}, \; e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix}.$

How to find the suitable coefficients to satisfy the initial conditions?

$$Y' = AY \quad , \quad Y(t_0) = Y_0 \text{ given}$$

$$Y = c_1 Y^{(1)} + \dots + c_n Y^{(n)}$$

$$= \underbrace{\begin{bmatrix} | & & | \\ Y^{(1)} & \cdots & Y^{(n)} \\ | & & | \end{bmatrix}}_{\substack{\Phi \text{ - fundamental} \\ \text{matrix}}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Substitute $t = t_0$:

$$Y_0 = \Phi(t_0) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \rightsquigarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi(t_0)^{-1} Y_0 .$$