Lecture 6

Saturday, September 11, 2021 10:52 PM

* Prayer

* Spiritual thoughts

* Compound interest: $\$1 \xrightarrow{dt} \$1 + r\,dt$

$$S(t) \xrightarrow{dt} S(t)(1+r\,dt) = S(t+dt)$$

$$\rightsquigarrow \underbrace{\frac{S(t+dt)-S(t)}{dt}}_{S'(t)} = rS(t)$$

Deposit rate $= k$:

$$S(t) \xrightarrow{dt} \underbrace{S(t)(1+r\,dt) + k\,dt}_{S(t+dt)}$$

$$\rightsquigarrow \underbrace{\frac{S(t+dt)-S(t)}{dt}}_{S'(t)} = rS(t) + k$$

* Existence and uniqueness theory:

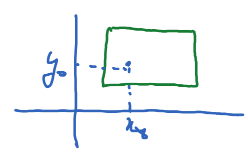
First order ODE of the form $y' = f(x,y)$.

In reality, x represents time and $y = y(x)$ is a quantity evolving with time. The ODE $y' = f(x,y)$ describes the law of change of y. Ideally, we want to be able to compute y at any given time x, assuming that we know the initial condition $y(x_0) = y_0$. (Laplace's demon) Although this thinking is not always satisfied or only satisfied under restricted conditions (short time of observation, unwanted physical interferences are completely removed, ...), it is an attractive idea to human minds and is a motivation for science and technology.

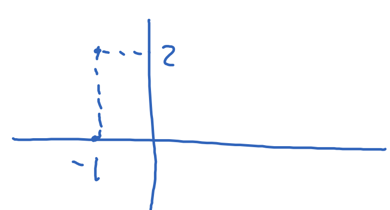
$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \qquad (*)$$

<u>Theorem</u>

If f and f_y are both continuous in a rectangle containing (x_0, y_0) then the problem $(*)$ has a unique sol. on an interval $(x_0 - h, x_0 + h)$ for some $h > 0$.



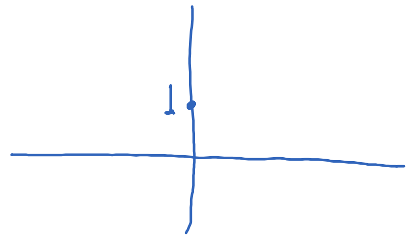
<u>Ex</u> $\begin{cases} y' = x + y \\ y(-1) = 2 \end{cases}$



The rectangle can be chosen arbitrarily.

Ex

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$



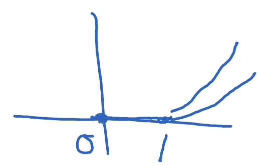
The rectangle can be chosen arbitrarily. But the solution only exists up to $x = 1$.

$$y = \frac{1}{1-x}$$ (solved by the method of separation of variables)

Ex

$$\begin{cases} y' = \dfrac{y}{(1-x)^2} \\ y(0) = 0 \end{cases} \rightsquigarrow \frac{dy}{y} = \frac{dx}{(1-x)^2}$$

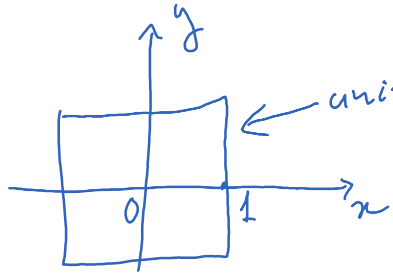
$$y = \begin{cases} C e^{\frac{1}{1-x}}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$



$$\ln|y| = \frac{1}{1-x} + k \rightsquigarrow |y| = e^{\frac{1}{1-x}} e^k$$

$$\rightsquigarrow y = C e^{\frac{1}{1-x}}$$

require $y \neq 0$



uniqueness is lost beyond $x = 1$

For linear ODE $y' = \underbrace{-p(x)y + g(x)}_{f(x,y)}$, we have

$\left[\begin{array}{l}\text{If } p \text{ and } g \text{ are continuous on an interval } I \text{ containing } x_0, \\ \text{then the sol. exists and is unique on } I.\end{array}\right]$

This seems to be a trivial statement because we have a formula for y.