Midterm II: Some problems for review

The second midterm exam will cover Sections 3.5-3.8, 4.1-4.3, 5.1-5.3, 6.1-6.4. The exam is a closed-book exam. Non-graphing calculators are allowed. The following power series will be provided on the exam:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k},$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

The table of Laplace transform on page 252 of the textbook will also be provided. Some problems you should know how to do:

- Solve ODEs of the form $ay'' + by' + cy = g(x)$ using the method undetermined coefficients and the method of variation of parameters.
- Solve a mass-spring problem with given mass, damping constant, spring constant, and external force. Decide if the vibration is overdamped, critically damped, underdamped or undamped.
- Solve higher order ODEs of the form $y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = g(t)$ using the method of "testing candidate" and undetermined coefficients.
- Solve an ODE by the power series method: find a recurrence relation or the first few coefficients.
- Compute Laplace transform / inverse Laplace of a function using the table. You need to know how to do partial fractional decomposition.
- Solve an ODE using Laplace transform.

In Problem 1 through 7, circle the correct answer. In the exam, you will not be asked to give explanations. But you should know the reason for your answers in this practice. In Problem 8 through 17, write out your answer carefully. Try not to skip any steps.

1. The general solution to the differential equation $y'' - 2y' - 3y = \cos t$ is of the form

$$y = c_1 e^{-t} + c_2 e^{3t} + A\cos t + B\sin t.$$

   a. True

   b. False

2. There is a solution to the differential equation $y'' - 4y' + 4y = e^{2t} + \cos t$ is of the form

   a. $y = Ae^{2t} + B\cos t + C\sin t$

   b. $y = Ate^{2t} + B\cos t + C\sin t$

   c. $y = At^2e^{2t} + B\cos t + C\sin t$

   d. $y = At^3e^{2t} + B\cos t + C\sin t$

3. A linear homogeneous ODE of order $n \geq 1$ with constant coefficients always has n linearly independent solutions.

   a. True

   b. False

4. A mass weighing 2 lb stretches a spring 2 in. Suppose the damping coefficient is 1 lb $\cdot$ s/ft. Denote by $u(t)$ the displacement of the mass from its equilibrium position, measured positive in the downward direction, at time t. Then u satisfies the equation

   a. $u'' + 16u' + 192u = 0$

   b. $u'' + 16u' + 192u = 32$

   c. $2u'' + u' + u = 0$

   d. $2u'' - u' + u = 0$

5. The motion described in Problem 3 is

   a. Overdamped

   b. Underdamped

   c. Critically damped

   d. Undamped

6. A mass-spring system with mass $m = 2$ and spring coefficient $k = 18$ is acted upon by a periodic external force $F = 2\sin(\omega t + \pi/6)$. Suppose the motion is undamped. Then resonance happens when ω is equal to

   a. 0

   b. 1

   c. 2

   d. 3

7. The differential equation $y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = g(t)$, where $p_1, \ldots, p_n, g$ are given functions with $g \neq 0$, is called a ______, ______ ODE.

   a. linear, nonhomogeneous

   b. linear, homogeneous

   c. nonlinear, nonhomogeneous

   d. nonlinear, homogeneous

8. Suppose that a linear, homogeneous ODE with constant coefficients has the characteristic polynomial $(r^2 + 1)^2(r + 1)$. What is the ODE? Determine the general solution.

9. Express the following function as a power series centered at 0. Determine the radius of convergence.

   (a) $f(x) = \frac{5}{3+2x^2}$

   (b) $g(x) = \ln(2x^2 + 3)$

10. Determine whether the following integral converges or diverges.

(a)

$$\int_0^\infty \frac{e^{-t^2+1}}{t}dt$$

(b)

$$\int_1^\infty \frac{\sin t + \cos t}{t^2}dt$$

(c)

$$\int_0^1 e^{-1/t}dt$$

(d)

$$\int_1^\infty \frac{t^2-5}{t^4+3}\sin(t)dt$$

11. Determine the interval of convergence of the power series

$$\sum_{k=1}^\infty \frac{(x-1)^k}{2^k k^2}.$$

12. Find the first three nonzero terms of a power series solution to the initial value problem

$$y' = 3y(1+xy), \quad y(0) = 1.$$

13. Solve the initial value problem

$$y'' + 3y' + 2y = \cos t, \quad y(0) = y'(0) = 1.$$

14. Solve the initial value problem

$$y'' - 2y' + y = 2e^{t/2}, \quad y(0) = y'(0) = 0.$$

15. Find the general solution of the ODE

$$y''' - 2y'' - y' + 2y = e^{4t} + \cos t.$$

16. Find the general solution of the ODE

$$y'' - 3y' + 2y = \sin(e^{-t}).$$

Hint: use variation of parameters.

17. Find the solution of the initial value problem

$$y'' - 3y' + 2y = g(t), \quad y(0) = 0, \; y'(0) = 1,$$

where

$$g(t) = \begin{cases} 1 & \text{if } \; 0 \le t < 2, \\ 0 & \text{if } \; \qquad t > 2. \end{cases}$$

Answer keys:

1. a
2. c
3. a
4. a
5. b
6. d
7. a
8. $y^{(5)} + y^{(4)} + 2y''' + 2y'' + y' + y = 0$
9. Part (a): $f(x) = \sum_{k=0}^{\infty} \frac{5}{3}(-1)^k \left(\frac{2}{3}\right)^k x^{2k}$. Radius of convergence is $\sqrt{3/2}$.

   Part (b): $g(x) = \ln 3 + \ln\left(\frac{2x^2}{3} + 1\right) = \ln 3 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left(\frac{2x^2}{3}\right)^k}{k} = \dots$ (simplify)
   Radius of convergence is $\sqrt{3/2}$.
10. (a) diverges by comparing with $1/t$. (b) converges by comparing with $2/t^2$. (c) converges by using the change of variable $u = 1/t$. (d) converges by comparing with $1/t^2$.
11. $(-1, 3)$
12. $y(x) = 1 + 3x + 6x^2 + \dots$
13. $\frac{1}{10}e^{-2t}(-16 + 25e^t + e^{2t}\cos t + 3e^{2t}\sin t)$
14. $4e^{t/2}\left(e^{t/2}t - 2e^{t/2} + 2\right)$
15. $c_1e^{-t} + c_2e^t + c_3e^{2t} + \frac{1}{30}\left(e^{4t} - 3\sin t + 6\cos t\right)$
16. $c_1e^t + c_2e^{2t} - e^{2t}\cos\left(e^{-t}\right)$
17. $$y(t) = \begin{cases} \frac{1}{2}\left(-4e^t + 3e^{2t} + 1\right) & \text{if} \quad 0 < t < 2 \\ \frac{1}{2}\mathrm{e}^{-4+t}\left(2\mathrm{e}^2 - 4\mathrm{e}^4 - \mathrm{e}^t + 3\mathrm{e}^{4+t}\right) & \text{if} \quad t > 2 \end{cases}$$