MATH 334, MIDTERM EXAM II, FALL 2021

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Name	Section # (Sec. 2: 11-12PM, Sec. 3: 12-1PM)

Instructions:

- This is a closed-book exam, 2 hours long. Non-graphing calculators are allowed.
- For Problems 13, 14, 15, you must solve the ODE using the specified methods. Make sure to show all necessary steps. Mysterious answers will receive little or no credit.
- Some formulae are provided on the next page.
- Do not discuss the exam with anyone before the exam window is closed (Nov 10-12).

Problem	Possible points	Earned points
1-12	24	
13	10	
14	10	
15	10	
Total	54	

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t^n e^{at}$, n a positive integer	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t) = \begin{cases} 0 & \text{if} \quad t < c, \\ 1 & \text{if} \quad t \geq c \end{cases}$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

- Variation of parameter: $y'' + py' + qy = g, \quad y = u_1y_1 + u_2y_2, \quad u_1' = -y_2g/W, \;\; u_2' = y_1g/W.$
- Some power series:

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \\
\ln(1+x) &= \sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k}, \\
\sin x &= \sum_{k=0}^{\infty} (-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!}, \\
\cos x &= \sum_{k=0}^{\infty} (-1)^{k}\frac{x^{2k}}{(2k)!}, \\
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k
\end{aligned}$$

Problem 1. (2 points) Which of the following is the best choice for the form of the particular solution when using the Method of Undetermined Coefficients to solve the differential equation

$$2y'' - y' - y = te^t$$

Here, "best" means that all terms required by this method are included and no extra terms are included.

A. $Ate^t + Be^t$
B. $At^2e^t + Bte^t$
C. Ate^t
D. At^2e^t

Problem 2. (2 points) Consider a mass-spring system with $m = 1$, $\gamma = 3$, $k = 2$ with external force $F(t) = \sin t$. The *transient part* of the solution is of the form

A. $c_1e^{-t} + c_2e^{-2t}$
B. $A\cos t + B\sin t$
C. $c_1e^{-t} + c_2e^{-2t} + A\cos t + B\sin t$
D. $c_1e^{-t} + c_2e^{-2t} + At\cos t + Bt\sin t$

Problem 3. (2 points) Consider a mass-spring system with external force $F(t) = \cos t$. Then, regardless of the initial conditions of the system, the vibration of the mass is always periodic.

A. True
B. False

Problem 4. (2 points) An ODE with characteristic polynomial $(r+1)^2(r^2+1)$ has a general solution

A. $c_1e^{-t} + c_2\cos t + c_3\sin t$
B. $c_1e^{-t} + c_2te^{-t} + c_3e^{-t}\cos t + c_4e^{-t}\sin t$
C. $c_1e^{-t} + c_2te^{-t} + c_3\cos t + c_4\sin t$
D. $c_1te^{-t} + c_2\cos t + c_3\sin t$

Problem 5. (2 points) Which of the following functions is NOT a solution of

$$y''' + 2y'' - y' - 2y = 0$$

A. $e^t + 2e^{-t}$
B. $2e^{-t} - e^{-2t}$
C. $3e^t$
D. $e^t + e^{2t}$

Problem 6. (2 points) The differential equation

$$(x^2+1)y''' + (\ln x)y'' + (\sqrt{1+x})y = 1$$

is sure to have a solution on the interval

A. $\mathbb{R}$
B. $(0,1)$
C. $(-1,0)$
D. $(0,\infty)$

Problem 7. (2 points) Which of the following power series represents the function $\frac{2}{3x-1}$?

A. $-2\sum_{k=0}^{\infty} 3^k x^k$

B. $2\sum_{k=0}^{\infty}(-3)^k x^k$

C. $-2\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k x^k$

D. $-2\sum_{k=0}^{\infty}\left(-\frac{2}{3}\right)^k x^k$

Problem 8. (2 points) The initial value problem

$$e^x y'' + xy = x, \quad y(0) = -1,\ y'(0) = 1$$

has a power series solution $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ The coefficients a_3 is equal to

A. $-1/3$
B. $1/6$
C. $1/3$
D. 0

Problem 9. (2 points) In solving the differential equation

$$y' - y = \frac{1}{1-x}$$

by the power series method, one has $y = \sum_{n=0}^{\infty} a_n x^n$ where the coefficients a_n's satisfy a recurrence relation

A. $a_{n+1} = \frac{1+a_n}{n-1}$

B. $a_{n+1} = \frac{1+a_n}{n+1}$

C. $a_n = \frac{1+a_{n+1}}{n+1}$

D. $a_n = \frac{1+a_{n+1}}{n-1}$

Problem 10. (2 points) For any function f and g, we have $\mathcal{L}\{fg\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

A. True
B. False

Problem 11. (2 points) Which of the following improper integrals diverges?

A. $\int_1^{\infty} e^{-t} dt$

B. $\int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt$

C. $\int_1^{\infty} \frac{\sin t}{t^2+1} dt$

D. $\int_0^{\infty} \frac{t^2}{t^3+1} dt$

Problem 12. (2 points) The Laplace transform of function $f(t) = 2e^{t+1}\cos 3t$ is equal to

A. $\frac{2(s-1)}{(s-1)^2+9} + 1$

B. $\frac{2s}{(s-1)^2+9}$

C. $\frac{2e(s-1)}{(s-1)^2+9}$

D. $\frac{2s}{s^2+9}$

Problem 13. (10 points) Solve the following initial value problem using the *Method of Undetermined Coefficients.*

$$y'' + 2y' + y = 4e^t, \; y(0) = 0, \; y'(0) = 1$$

$y = y_c + y_p$

* Find y_c: characteristic eq. $r^2+2r+1=0 \longrightarrow r=-1$ (double root)

$$y_c = e^{-t}(c_1 + tc_2)$$

* Find y_p: guess $y_p = Ae^t$.

$$y_p'' + 2y_p' + y_p = 4Ae^t \rightsquigarrow A = 1.$$

Thus, $y = e^{-t}(c_1 + tc_2) + e^t$

* Use initial conditions to determine c_1, c_2:

$$0 = y(0) = c_1 + 1 \rightsquigarrow c_1 = -1$$

$$y' = e^{-t}c_2 - e^{-t}(c_1 + tc_2) + e^t \rightsquigarrow \underbrace{y'(0)}_{1} = c_2 - c_1 + 1 = c_2 + 2$$

$$\rightsquigarrow c_2 = -1$$

In conclusion, $y = e^{-t}(-1-t) + e^t$

Problem 14. (10 points) Find the general solution of the following differential equation using the *Variation of Parameter Method.*

$$y'' - 3y' + 2y = te^t.$$

Characteristic eq. $r^2 - 3r + 2 = 0 \leadsto r_1 = 1, \; r_2 = 2$

$\leadsto y_1 = e^t, \; y_2 = e^{2t}$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$$

$$u_1' = -\frac{y_2 g}{W} = -\frac{e^{2t} t e^t}{e^{3t}} = -t \leadsto u_1 = -\frac{t^2}{2} + c_1$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^t t e^t}{e^{3t}} = te^{-t} \leadsto u_2 = \int te^{-t} dt$$

Integration by part: $u = t, \; dv = e^{-t} dt$

$du = dt, \; v = -e^{-t}$

$$u_2 = \int te^{-t} dt = -te^{-t} + \int e^{-t} dt = -te^{-t} - e^{-t} + c_2$$

Thus,

$$y = u_1 y_1 + u_2 y_2 = \left(-\frac{t^2}{2} + c_1\right) e^t + \left(-te^{-t} - e^{-t} + c_2\right) e^{2t}$$

Problem 15. (10 points) Solve the following initial value problem using *Laplace transform*.

$$y' - y = g(t), \quad y(0) = 0,$$

$$g(t) = \begin{cases} -1 & \text{if } \; t < 1, \\ 2 & \text{if } \; t > 1. \end{cases} = -1 + 3u_1(t)$$

Take Laplace transform of both sides of $y' - y = -1 + 3u_1(t)$:

$$sY - \underbrace{y(0)}_{0} - Y = -\frac{1}{s} + 3\,\frac{e^{-s}}{s}$$

$$\rightsquigarrow \quad Y = -\underbrace{\frac{1}{s(s-1)}}_{F(s)} + 3e^{-s}\underbrace{\frac{1}{s(s-1)}}_{F(s)}$$

$$\rightsquigarrow \quad y(t) = \mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\{F\} + 3\,\mathcal{L}^{-1}\{e^{-s}F\} \qquad (*)$$

Fractional decomposition:

$$F(s) = \frac{1}{s-1} - \frac{1}{s} \quad \rightsquigarrow \quad \mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= e^{t} - 1 = f(t)$$

From $(*)$: $\quad y(t) = -f(t) + 3\,u_1(t)\,f(t-1)$

$$y(t) = -e^{t} + 1 + 3u_1(t)\left(e^{t-1} - 1\right)$$