* Step by step solution to the example

$$y'' + ty' - 2y = 0 \qquad (*)$$

knowing one solution $y_1 = t^2+1$.

Write $y = (t^2+1)v$ where v is a function of t to be determined.
Substitute this function into $(*)$ and reduce:

$$(t^2+1)v'' + (t^3+5t)v' = 0$$

Put $u = v'$. Then

$$(t^2+1)u' + (t^3+5t)u = 0$$

$$\rightsquigarrow \quad u' + \underbrace{\frac{t^3+5t}{t^2+1}}_{p(t)} u = 0 \qquad (**)$$

One can find u by the integrating factor method:

$$\int p(t)\,dt = \int \left(t + \frac{4t}{t^2+1}\right) dt = \frac{t^2}{2} + 2\ln(t^2+1).$$

Integrating factor:

$$\mu(t) = \exp\left(\frac{t^2}{2} + 2\ln(t^2+1)\right) = e^{\frac{t^2}{2}} e^{2\ln(t^2+1)}$$

$$= e^{\frac{t^2}{2}}(t^2+1)^2$$

Multiplying both sides of $(**)$ by $\mu(t)$, we get

$$(\mu u)' = 0 \quad \rightsquigarrow \quad u = \frac{c}{\mu} = c\,e^{-t^2/2}(t^2+1)^{-2}.$$

To get v, we integrate u:

$$v = c\int e^{-\frac{t^2}{2}}(t^2+1)^{-2}\,dt$$

$$= c_1 \int_0^t e^{-\frac{s^2}{2}}(s^2+1)^{-2}\,ds + c_2$$

Therefore, all solutions to $(*)$ are

$$y = (t^2+1)v = c_1(t^2+1)\int_0^t e^{-s^2/2}(s^2+1)^{-2}\,ds + c_2(t^2+1)$$