Sketch region after a change of variables

After the change of variables $(x, y) \to (u, v)$, the region R of (x, y) is transformed into a region R' of (u, v). The purpose of this note is to sketch the region R' using Mathematica. Example 1 applies for 2D, Example 2 and Example 3 for both 2D and 3D.

Example 1:

Let $R = \{(x, y) : (x-2)^2 + (y-2)^2 \le 1/4\}$, i.e. the disk of radius 1/2 centered at (2, 2), and consider the change of variables $x = u^2 v, \ \ y = u v$. To sketch the region R' of (u, v), we first solve for u and v in terms of x and y.

In[ ]:=
```
Solve[x == u^2 v && y == u v, {u, v}]
```

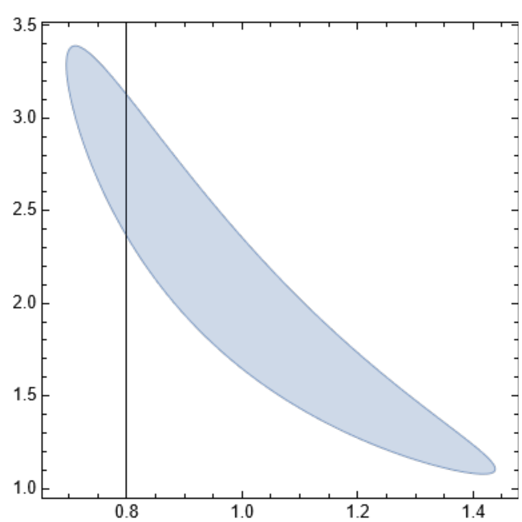
Out[ ]= $\left\{\left\{u \to \frac{x}{y}, v \to \frac{y^2}{x}\right\}\right\}$

Note: if the change of variables is already given in the form $u = u(x, y)$, $v = v(x, y)$, then the above step is not necessary.

Now that we know $u = x/y$ and $v = y^2/x$, we can draw the region R' using the command **ParametricPlot**.

In[ ]:=
```
R = ImplicitRegion[(x - 2)^2 + (y - 2)^2 ≤ 1 / 4, {x, y}];
ParametricPlot[{x / y, y^2 / x}, {x, y} ∈ R, AspectRatio → 1]
```

Out[ ]=

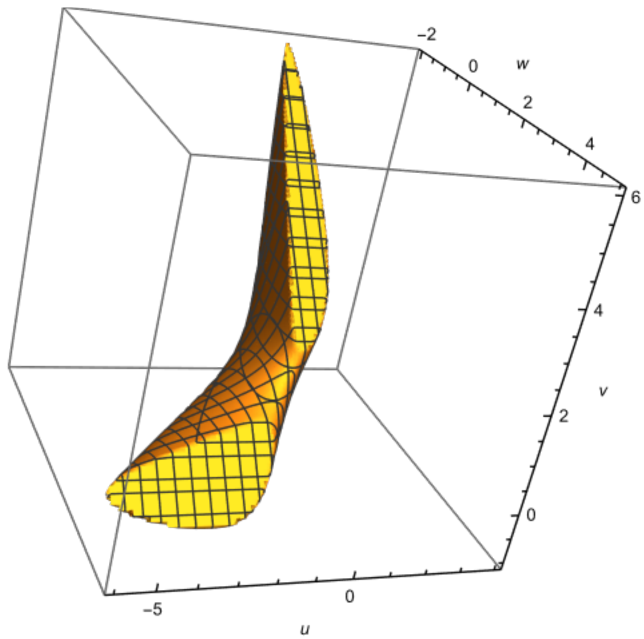


Example 2:

Let $R = \{(x, y, z) : x^2 + y^2 \le z \le 4\}$, i.e. the solid bounded by the elliptic paraboloid $z = x^2 + y^2$ and the plane $z = 4$, and consider the change of variables $x = u v, \ \ y = u + w, \ z = v + w$. To sketch the region R' of (u, v, w), we use the command **RegionPlot3D**.

In[∘]:=
```
x = u v;
y = u + w;
z = v + w;
RegionPlot3D[x^2 + y^2 ≤ z ≤ 4, {u, -6, 4},
 {v, -1, 6}, {w, -2, 5}, PlotPoints → 100, AxesLabel → {u, v, w}]
```

Out[∘]=



Note: *if the change of variables is given in the form* $u = u(x, y)$, $v = v(x, y)$, *then we solve for* x, y, z *in terms of* u, v, w *before using RegionPlot3D. For example:* $u = x + y$, $v = y + z$, $w = z + x$.

In[∘]:=
```
Clear[x, y, z];
Solve[u == x + y && v == y + z && w == z + x, {x, y, z}]
```

Out[∘]= $\left\{\left\{x \to \frac{1}{2}(u - v + w),\ y \to \frac{1}{2}(u + v - w),\ z \to \frac{1}{2}(-u + v + w)\right\}\right\}$

Example 3: *(for Mathematica 10 or later)*

In general, one can plot the image or pre-image of a region under a transformation using the command **TransformedRegion** or **InverseTransformedRegion**, respectively. Let R be a region described in spherical coordinates as follows:

$$R = \{(x, y, z) : 0 \le \rho \le \phi,\ 0 \le \phi \le \pi/2,\ 0 \le \theta \le \pi\}$$

We see that R is the image of the region

$$R' = \{(\rho, \theta, \phi) : 0 \le \rho \le \phi,\ 0 \le \phi \le \pi/2,\ 0 \le \theta \le \pi\}$$

under the transformation $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)$.

In[ ]:=
```
R' = ImplicitRegion[0 ≤ ρ ≤ ϕ && 0 ≤ ϕ ≤ π / 2 && 0 ≤ θ ≤ π, {ρ, θ, ϕ}];
f[ρ_, θ_, ϕ_] := {ρ Sin[ϕ] Cos[θ], ρ Sin[ϕ] Sin[θ], ρ Cos[ϕ]};
R = TransformedRegion[R', f];
Region[R, Axes → True, AxesLabel → {x, y, z}]
```

Out[ ]=