Final exam: Some problems for review

The exam will take place in our regular classroom TMCB 112 from 5 PM to 6:50 PM on Wednesday June 15, 2022. It is a closed-book exam, covering Sections 14.7-16.7. No calculators are allowed. The following formulae will be provided on the exam:

$$D = f_{xx}f_{yy} - f_{xy}^2 \text{ (for Second Derivative test of function } f(x,y))$$

$$\begin{cases} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \ , \quad J = \rho^2 \sin\phi \\ z &= \rho\cos\phi \end{cases}$$

$$\iint_S f dS = \iint_R f(r(u,v))|r_u \times r_v| dA$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R F(r(u,v)) \cdot (\pm r_u \times r_v) dA$$

You should review the homework problems, especially the odd problems (solution in the back of the textbook), and the examples given in class. Also, take a look at the practice exam posted on Learning Suite. It is always a good idea to study with someone. You should know how to solve following problems:

- Find local/absolute min/max and saddle points of a function with or without constraints.
- Use Second Derivative test to classify the critical points into local min, local max, saddle point.
- Evaluate double/triple integral.
- Find the Jacobian of a change of variables.
- Use spherical and cylindrical coordinates to evaluate triple integrals.
- Sketch by hand a vector field.
- Check if a vector field is conservative. If it is, find the potential function.
- Find curl and divergence of a vector field. Interpret them on the picture.
- Evaluate line integral using: parametrization, fundamental theorem of Calculus, Green's theorem.
- Evaluate surface integral using parametrization.

Some problems for practice:

1. Let S be the portion of the cylinder $y = x^2$, $0 \le x \le 1$, cut by the plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Find $\iint_S x dS$.

2. Evaluate the flux of the vector field $F(x,y,z) = (4x + z, -x - y, 2y + z)$ across the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,4)$ with downward orientation.

3. Evaluate the integral $\int_C (x^2 + y^2)dx + (x^2 - y^2)dy$ where C is the closed triangular curve with vertices at $(0,0)$, $(2,1)$, $(0,1)$ oriented in that order.

4. Let S be the cylinder $y^2 + z^2 = 5$. Find the equation of the tangent plane to S at the point $(3,2,1)$.

5. Compute $\iint_D \frac{y}{1+x^2} dA$ where D is the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 1$.

6. Compute $\iiint_E z dV$ where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

7. Find the maximum and minimum values of $f(x, y) = x^2 + y^2 + 4x - 4y$ in the disk $x^2 + y^2 \leq 9$.

8. Find all critical points of the function $f(x, y) = x^3 + x^2 + xy + y^2$ and classify them using the Second Derivative test.

9. A function $f(x, y)$ has at most two critical points. True or false?

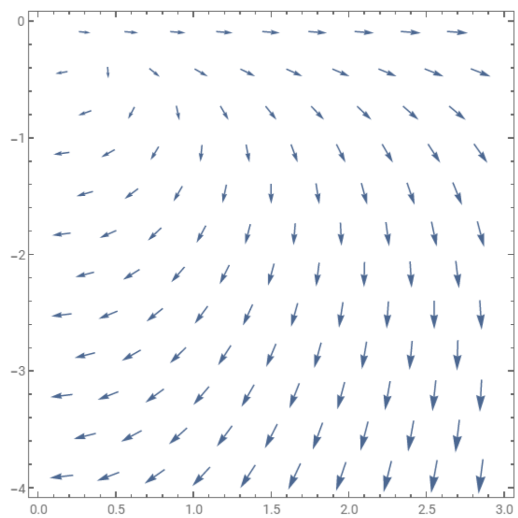
10. Let $F = (3x^2 + 2xy,\ x^2 + 2y)$. Is F a conservative vector field? If so, what is a potential function of F?

11. Let $F = (6xy + yz,\ 3x^2 + xz,\ xy + 2z)$. Is F a conservative vector field? If so, what is a potential function of F?

12. If F is a vector field then $\operatorname{div} F$ is also a vector field. True or false?

13. If F is a vector field then $\operatorname{curl} F$ is also a vector field. True or false?

14. A 2D vector field F is visualized as follows. What can you tell about the sign of $\operatorname{div} F$ and $\operatorname{curl} F$ at point $(1, -2)$?



15. Convert the Cartesian coordinates $(x, y, z) = (-2, 2, 2\sqrt{6})$ into spherical coordinates.

16. Convert the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/3, \pi/2)$ into Cartesian coordinates.

17. Describe the solid cut from the unit ball $x^2 + y^2 + z^2 \leq 1$ by the cone $z = \sqrt{x^2 + y^2}$ in spherical coordinate. In other words, what are the ranges for ρ, θ, and ϕ ?

18. Evaluate $\iiint_E (z + 1)dV$ where E is the solid enclosed by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

19. Use the change of variables $x = u^2$, $y = v^2$ to find the area of the region bounded by the curve $\sqrt{x} + \sqrt{y} = 1$, the x-axis, and the y-axis.

Answer keys:

1) $\frac{1}{140}\left(50\sqrt{5}+1\right)$
2) $-34/3$
3) Green's theorem: $\int_0^2 \int_{x/2}^1 (2x-2y)dydx = 0$
4) $2y+z-5=0$
5) $\frac{1}{4}\ln 2$
6) $8\pi/3$
7) min$=-8$, max$=9+12\sqrt{2}$
8) Two critical points: $(-1/2, 1/4)$ (saddle point) and $(0,0)$ (local minimum is attained)
9) False
10) Yes, because $\operatorname{curl} F = Q_x - P_y = 0$. Potential function $f(x,y) = x^3 + x^2y + y^2$.
11) Yes, because $\operatorname{curl} F = 0$. Potential function $f(x,y,z) = 3x^2y + xyz + z^2$.
12) False
13) It depends. If F is a 3D vector field, then yes. If F is a 2D vector field, then no.
14) At $(1,-2)$, the divergence is positive, and the curl is negative.
15) $(\rho,\theta,\phi) = (4\sqrt{2}, 3\pi/4, \pi/6)$
16) $(x,y,z) = (1,\sqrt{3},0)$
17) $0\le\rho\le 1$, $0\le\phi\le\pi/6$, $0\le\theta\le 2\pi$
18) 8π
19) $1/6$