Lecture 11

Friday, May 20, 2022 12:13 PM

* Prayer
* Spiritual thought

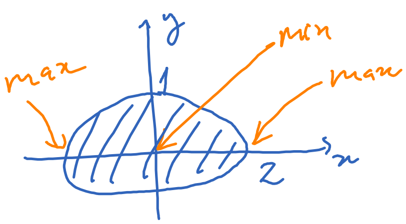
Recall: gradient is perpendicular to level set.

Plot using Mathematica: $f(x,y) = x^2 + xy + y^2$

* Application of derivatives in optimization problem.

Problem 1: Find min/max of $f(x,y)$ in a region D.

Ex $f(x,y) = x^2 + y^2$, (x,y) lies inside the ellipse.



Ex $f(x,y) = x^2 + y^2$, $(x,y) \in \mathbb{R}^2$

$\min f = 0$, attained at $(0,0)$

$\max f$ doesn't exist.

Observation:

- min/max may or may not exist, depending on the region D.
- If f is continuous on D, and D is closed, the min & max exist.

How to determine min/max?

Min/max can be attained inside D or on the boundary of D.

Calc I: To find min/max of f on $[a,b]$, we find all the critical points of f in (a,b). The we compare the values of f at those critical points with the values of f at a and b.

We will follow a similar procedure for multivariable functions.

(1) Find all $\underbrace{\text{critical points}}_{\nabla f(x,y)=0}$ of f inside D.

(2) Find min/max of f on the boundary of D

(3) Compare these values with the values of f at the critical points.

Question: why is min/max not attained at any point inside D other than the critical point?

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2$ on $D =$ circle of radius $\frac{3}{2}$.

- Find critical points:

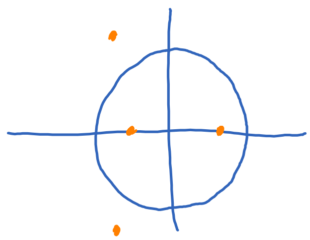
$$\nabla f = 0 \rightsquigarrow \begin{cases} 3x^2 + y^2 - 6 = 0 \\ 2xy + 2y = 0 \rightsquigarrow 2y(x+1) = 0 \rightsquigarrow \underbrace{y=0}_{A} \text{ or } \underbrace{x=-1}_{B} \end{cases}$$

Case (A): $y=0 \rightsquigarrow x^2=2 \rightsquigarrow x=\pm\sqrt{2}$

Two critical points $(\sqrt{2},0)$, $(-\sqrt{2},0)$.

Case (B): $x=-1 \rightsquigarrow y^2=3 \rightsquigarrow y=\pm\sqrt{3}$

Two critical points $(-1,\sqrt{3})$, $(-1,-\sqrt{3})$.



The only critical points inside the region are $(\pm\sqrt{2},0)$.

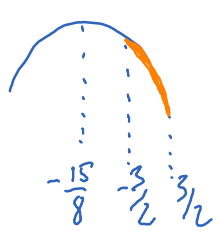
- Find min/max of f on the boundary of D:

$$x^2+y^2=\left(\frac{3}{2}\right)^2 \longrightarrow y^2=\frac{9}{4}-x^2$$

$$f(x,y)=x^3+xy^2-6x+y^2 = x^3+x\left(\frac{9}{4}-x^2\right)-6x+\left(\frac{9}{4}-x^2\right)$$

$$=-x^2-\frac{15}{4}x+\frac{9}{4}$$

We find min/max of this function on $\left[-\frac{3}{2},\frac{3}{2}\right]$.



$$\min_{bd} f = f\left(\frac{3}{2},0\right)$$

$$\max_{bd} f = f\left(-\frac{3}{2},0\right)$$

Conclusion:

$$\min_D f = \min\left\{f(\sqrt{2},0),\ f(-\sqrt{2},0),\ f\left(\frac{3}{2},0\right)\right\} = \cdots$$

$$\max_D f = \max\left\{f(\sqrt{2},0),\ f(-\sqrt{2},0),\ f\left(-\frac{3}{2},0\right)\right\} = \cdots$$

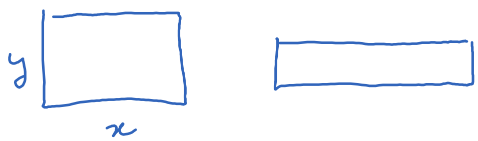
* Optimization problem under constraints:

$$\begin{cases} f(x,y) \longrightarrow \text{min/max} \\ g(x,y) = 0 \leftarrow \text{this is called a constraint.} \end{cases}$$

Ex

$$\begin{cases} f(x,y) = x^3 + xy^2 - 6x + y^2 \longrightarrow \text{min/max} \\ g(x,y) = x^2 + y^2 - \frac{9}{4} = 0 \end{cases}$$

Ex



Fencing a garden with 100ft of wire fence.

How to make the area maximum?

$xy \longrightarrow \max$

under constraint $2x + 2y - 100 = 0$

Lagrange multiplier method

$$\begin{cases} f(x,y) \longrightarrow \text{min/max} \\ g(x,y) = 0 \end{cases}$$

① Solve λ, x, y from the system

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

② Compare the values of f at those x, y.

Ex

$$f(x,y) = xy$$

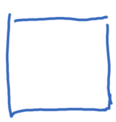
$$g(x,y) = 2x + 2y - 100$$

First, solve λ, x, y from

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \rightsquigarrow \begin{cases} (y, x) = \lambda(2,2) \\ 2x + 2y - 100 = 0 \end{cases} \rightsquigarrow \begin{cases} x = y = 2\lambda \\ 4\lambda + 4\lambda - 100 = 0 \end{cases}$$

$$\rightsquigarrow \begin{cases} x = y = 25 \\ \lambda = \frac{25}{2} \end{cases}$$

$\max\limits_{g=0} f$ is attained when $x = y = 25$.

 garden has a square shape.