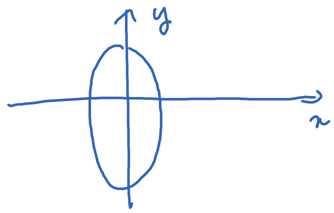
Lecture 12

Saturday, May 21, 2022 11:03 PM

* Prayer
* Spiritual thought

Optimization problem with constraints

$f(x,y) = 4x^2+2y^2+3x+3y \longrightarrow$ min/max in the ellipse $2x^2+y^2 \le 1$.



Step 1: find critical points inside D.

Step 2: find min/max on the boundary of D.

This is an optimization problem under constraint:

$$\begin{cases} f(x,y) = 4x^2+2y^2+3x+3y \longrightarrow \text{min/max} \\ g(x,y) = 2x^2+y^2 = 1 \end{cases}$$

Notice: $f(x,y) = 2(\underbrace{2x^2+y^2}_{=1}) + 3(x+y) = 2 + 3\underbrace{(x+y)}_{h(x,y)}$

$$\begin{cases} h(x,y) \rightarrow \text{min/max} \\ g(x,y) = 0 \end{cases}$$

Lagrange multiplier method:

$$\begin{cases} \nabla h = \lambda \nabla g \\ g = 0 \end{cases} \rightsquigarrow \begin{cases} (1,1) = \lambda(4x, 2y) \\ 2x^2 + y^2 = 1 \end{cases}$$

$$\rightsquigarrow \begin{cases} x = \frac{1}{4\lambda}, \; y = \frac{1}{2\lambda} \\ 2\left(\frac{1}{4\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \end{cases}$$

$$\rightsquigarrow \frac{3}{8\lambda^2} = 1 \rightsquigarrow \lambda = \pm\sqrt{\frac{8}{3}}$$

If $\lambda = \sqrt{\frac{8}{3}}$, then $x = \frac{\sqrt{3}}{4\sqrt{8}}$ and $y = \frac{\sqrt{3}}{2\sqrt{8}}$. (1)

If $\lambda = -\sqrt{\frac{8}{3}}$, then $x = \frac{-\sqrt{3}}{4\sqrt{8}}$ and $y = \frac{\sqrt{3}}{2\sqrt{8}}$. (2)

h is max in scenario (1), min in scenario (2).

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* Classification of critical points:

Calc I: $f'(x) = 0$

$$f''(x) \begin{cases} > 0 \longrightarrow \text{local min} \\ < 0 \longrightarrow \text{local max} \\ = 0 \longrightarrow \text{inconclusive (inflection point)} \end{cases}$$

This is called Second Derivative Test.

Multivariable calculus:

$$D = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Hessian matrix

If $f_{xx} > 0$ and $D > 0$ then local min.

If $f_{xx} < 0$ and $D > 0$ then local max.

If $f_{xx} = 0$ and $D < 0$ then saddle point.

If $D < 0$ then saddle point.

If $D = 0$ then inconclusive.

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2$

has 4 critical points: $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(-1, \sqrt{3})$, $(-1, -\sqrt{3})$.

Classify each critical point.

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* For functions with 3 variables $f(x,y,z)$, we write the Hessian matrix

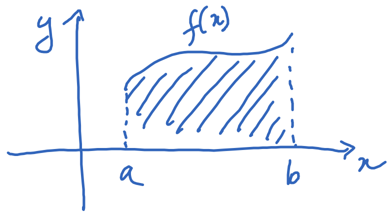
$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

If H is positively definite (i.e. all eigenvalues > 0) then local min.

If H is negatively definite (" " < 0 ") " max.

Integral:

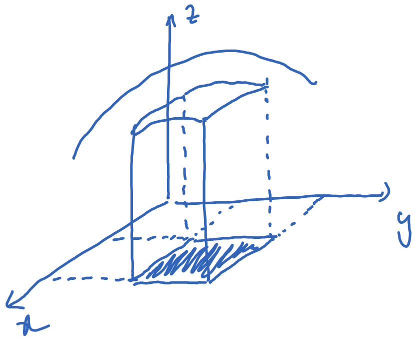
Calc I:



$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx$ = area under the curve

and above the interval $[a,b]$.

Multivariable calculus:



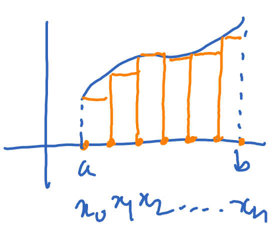
$\iint_{[a,b]\times[c,d]} f(x,y)\,dA$ = volume under the surface (graph of f)

and above the rectangle $[a,b]\times[c,d]$.

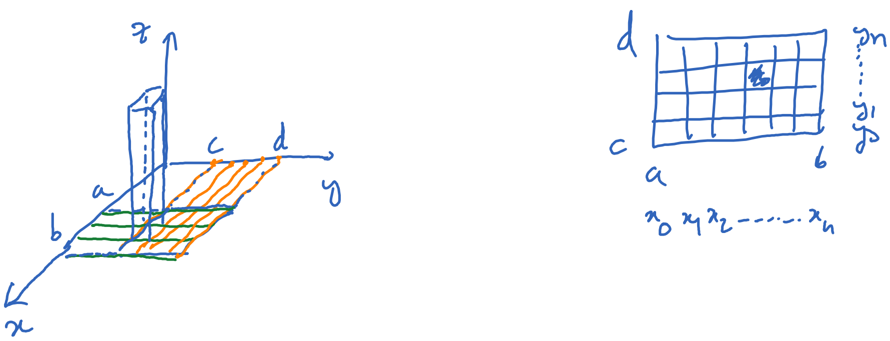
This is called a double integral

How to calculate a double integral?

Recall: how to calculate $\int_a^b f(x)\,dx$?



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n-1} f(x_k)\,\Delta x}_{\text{Riemann sum}}$$



volume of the (k,j)-prism is $f(x_k, y_j)\,\Delta A$.

Total volume $\approx \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k, y_j)\, \Delta A$

$$\iint_{[a,b]\times[c,d]} f(x,y)\, dA = \lim_{n\to\infty} \sum_{k=0}^{n-1} \underbrace{\sum_{j=0}^{n-1} f(x_k, y_j)\, \Delta A}_{\int_c^d f(x,y)\, dy}$$

$$\underbrace{\qquad\qquad}_{\int_a^b \int_c^d f(x,y)\, dy\, dx}$$

Therefore,

$$\iint_{[a,b]\times[c,d]} f(x,y)\, dA = \underbrace{\int_a^b \int_c^d f(x,y)\, dy\, dx}$$

freeze x, integrate wrt y,

then integrate wrt x. This is called

an <u>iterated integral</u>.