# Lecture 14

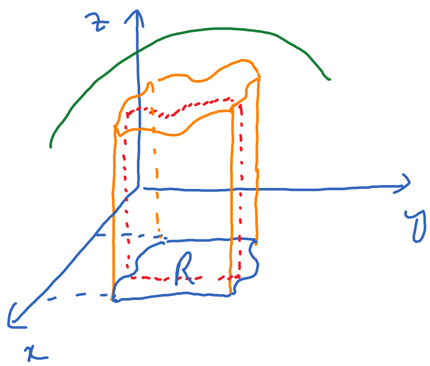
Thursday, May 26, 2022 9:27 PM

* Prayer

* Spiritual thought

* Recall double integral over a region $R = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$:

$$\iint_R f(x,y)\,dA = \text{volume under graph of } f \text{ and above } R.$$

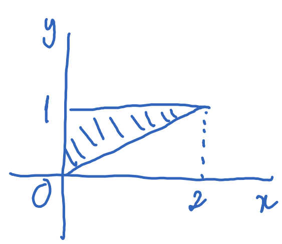



$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

Note: $\displaystyle\int_{g(x)}^{h(x)} f(x,y)\,dy$ = area of the x-intersection of the solid $= S(x)$

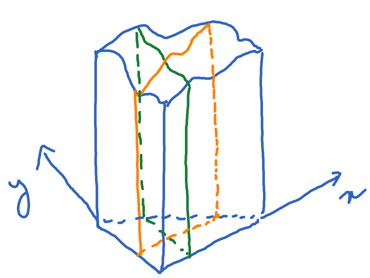
$$\text{volume} = \int_a^b S(x)\,dx.$$

Ex:

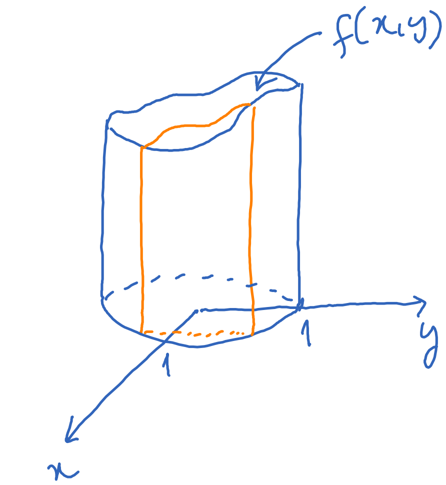



$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x/2}^{1} f(x,y)\,dy\,dx$$

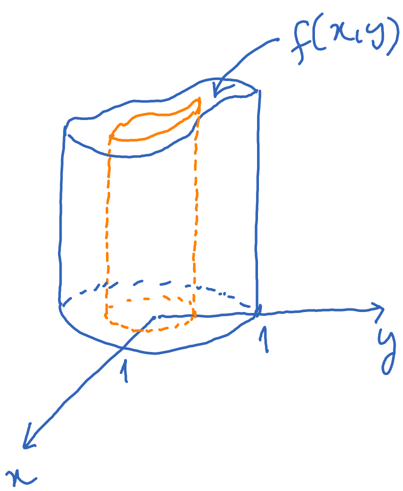
$$= \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy$$



* Double integral over a disk:



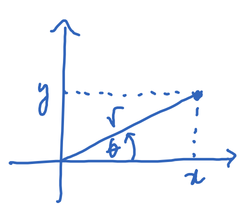

$$\iint_D f(x,y)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$



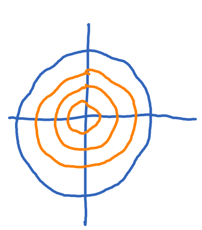

← This is another way of scanning the solid.

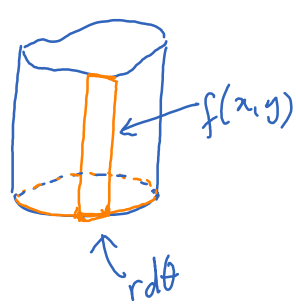
We use polar coordinates.

Recall:




$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r > 0,\quad 0 \le \theta < 2\pi$$






area of $S(r)$ is

$$\int_0^{2\pi} f(x,y)\, r\,d\theta$$

Volume of the solid is:

$$V = \int_0^1 S(r)\,dr = \int_0^1 \int_0^{2\pi} f(x,y)\, r\, d\theta\, dr$$

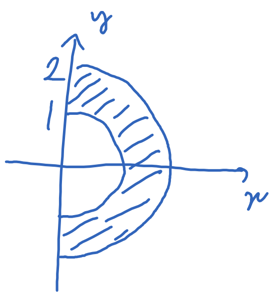
<u>Observation:</u>

If a region $R$ is described in polar coordinates as

$R = \{ (x,y) \mid a \le r \le b,\ c \le \theta \le d \}$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\, r\, d\theta\, dr$$

<u>Ex:</u>

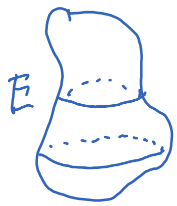



$$\iint_R x\,dA = \int_1^2 \int_{-\pi/2}^{\pi/2} r\cos\theta\; r\,d\theta\,dr$$

$$= \underbrace{\int_1^2 r^2\,dr}_{\frac{7}{3}} \; \underbrace{\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta}_{2} = \frac{14}{3}.$$

<u>Triple integral</u>

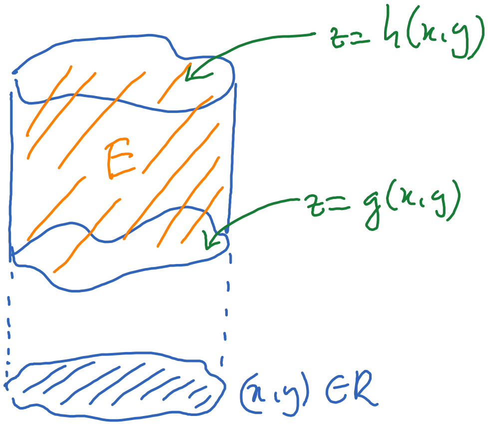
$$\iiint_{[a,b]\times[c,d]\times[h,k]} f(x,y,z)\,dV = \int_a^b \int_c^d \int_h^k f(x,y,z)\,dz\,dy\,dx.$$




$$\iiint_E f(x,y,z)\,dV = \text{mass of solid } E \text{ with mass density } f(x,y,z).$$

$$\text{vol}(E) = \text{mass of } E \text{ when mass density is } 1$$

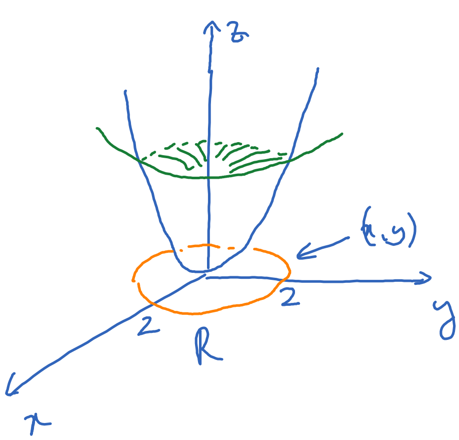
$$= \iiint_E 1\,dV$$




$$\iiint_E f(x,y,z)\,dV = \iint_R \int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\,dA$$

**Ex** Find $\iiint_E (xy+z^2)\,dV$ where $E$ is the solid between two surfaces

$z = x^2+y^2$ and $z = 2 + \frac{1}{2}(x^2+y^2)$.




Intersection of two surfaces:

$$\begin{cases} z = x^2+y^2 \\ x^2+y^2 = 2+\frac{1}{2}(x^2+y^2) \end{cases} \rightsquigarrow \begin{cases} z = 4 \\ x^2+y^2 = 4 \end{cases}$$

$$\iiint_E (xy+z^2)\,dV = \iint_R \int_{x^2+y^2}^{2+\frac{1}{2}(x^2+y^2)} (xy+z^2)\,dz\,dA$$

$$= \iint_R \left[ xy\left(2-\tfrac{1}{2}(x^2+y^2)\right) + \frac{\left(2+\frac{1}{2}(x^2+y^2)\right)^3}{3} - \frac{(x^2+y^2)^3}{3} \right] dA$$

$= \dots$ use polar coordinates

$$= \frac{56\pi}{3}.$$

On Mathematica: