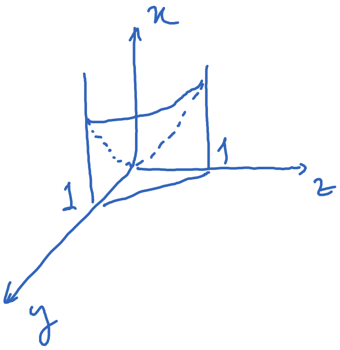
Lecture 15

Wednesday, June 1, 2022 12:40 PM

* Prayer

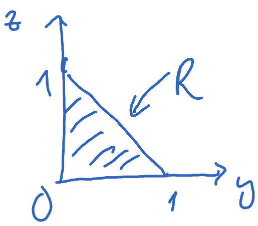
* Spiritual thought

* <u>Another example of triple integral:</u>



Find $\iiint_E (z+y)\,dV$ where E is the solid bounded by the surfaces $x = y^2+z^2$, $y+z=1$, $y=0$, $z=0$.

$$E = \{(x,y,z) \mid (y,z)\in R,\ 0 \le x \le y^2+z^2\}$$



$$\iiint_E (z+y)\,dV = \iint_R \int_0^{y^2+z^2} (z+y)\,dx\,dA$$

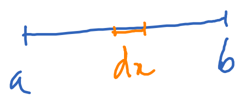
$$= \iint_R (z+y)(y^2+z^2)\,dA$$

$$= \int_0^1 \int_0^{1-y} (z+y)(y^2+z^2)\,dz\,dy = \ldots$$

<u>Change of variables:</u>

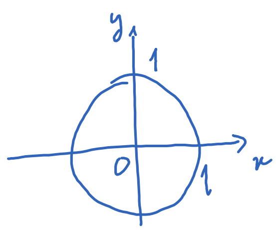
Calc I: $\displaystyle\int_{[0,2]} x\sin(x^2)\,dx = \int_0^4 \sqrt{t}\,\sin t \ldots\ dt$

In Calc I, we change variables to get a simple integrand.

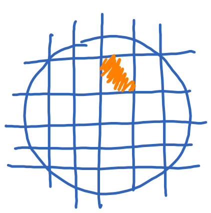


$$\int_a^b f(x)\,dx \qquad\qquad \int_c^d f(x(t))\,\underbrace{x'(t)}_{\text{stretching factor}}\,dt$$

In multivariable calculus, we change variables mostly due to the region we integrate over, not because of the integrand.



$$\iint_D x^2\,dA \xrightarrow{(x,y)\to(r,\theta)} \int_0^1\int_0^{2\pi} (r\cos\theta)^2\,\underbrace{r}_{\text{stretching factor}}\,d\theta\,dr$$



$dA = dx\,dy$

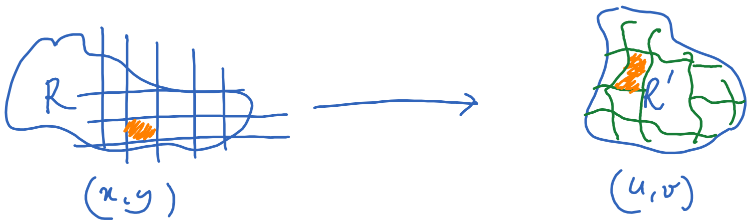
$$dA = \frac{(r+dr)^2 d\theta}{2} - \frac{r^2 d\theta}{2}$$

$$= \frac{r^2 d\theta + 2r\,dr\,d\theta + (dr)^2 d\theta - r^2 d\theta}{2}$$

$$= r\,dr\,d\theta + \underbrace{(dr)^2 d\theta}_{\approx 0 \text{ (too small)}}$$

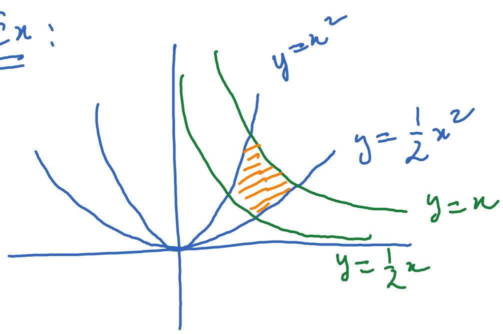
In general,

$$\iint_R f(x,y)\,dA \overset{(x,y)\to(u,v)}{=\!=\!=} \iint_{R'} f(x(u,v), y(u,v))\, J\, dA'$$



$$dA = J\,dA', \quad J = \left|\det \underbrace{\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}}_{\text{Jacobian matrix}}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| : \text{Jacobian of the transformation}$$

Ex:



$$\iint_R xy\,dA = ?$$

$$\begin{cases} y = ux^2 \\ y = \dfrac{1}{vx} \end{cases} \longrightarrow \begin{cases} y = \dfrac{u}{v^2}\dfrac{1}{y^2} \\ x = \dfrac{1}{yv} \end{cases}$$

$$\longrightarrow \begin{cases} y = u^{1/3} v^{-2/3} \\ x = y^{-1} v^{-1} = u^{-1/3} v^{2/3} v^{-1} = u^{-1/3} v^{-1/3} \end{cases}$$

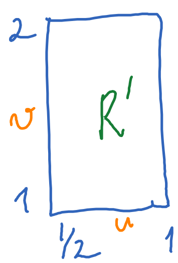
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} -\frac{1}{3}u^{-4/3}v^{-1/3} & -\frac{1}{3}u^{-1/3}v^{-4/3} \\ \frac{1}{3}u^{-2/3}v^{-2/3} & -\frac{2}{3}u^{1/3}v^{-5/3} \end{pmatrix}$$

$$= \frac{2}{9}u^{-1}v^{-2} + \frac{1}{9}u^{-1}v^{-2}$$

$$= \frac{1}{3}u^{-1}v^{-2}$$

$$\iint_R xy\,dA = \iint_{R'} u^{-1/3}v^{-1/3}\,u^{1/3}v^{-2/3}\,\frac{1}{3}u^{-1}v^{-2}\,dA'$$

$$= \iint_{R'} \frac{1}{3}u^{-1}v^{-3}\,dA' = \frac{1}{3}\int_{1/2}^{1} u^{-1}\,du \int_{1}^{2} v^{-3}\,dv = \frac{\ln 2}{8}$$



<u>Triple integral</u>

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,J\,dV'$$

where $J = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

Ex: cylindrical coordinates

$$(x,y,z) \longrightarrow (r,\theta,z)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \rightsquigarrow J = r$$

One can use Mathematica to sketch the new region after a change of variables. See the supplement material posted on the course website.