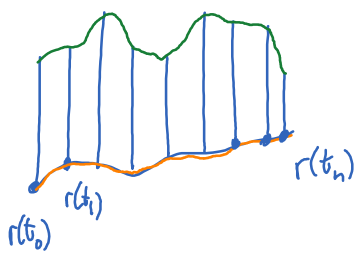
Lecture 17

Monday, June 6, 2022 11:11 AM

* Prayer

* Spiritual thought

Line integral



$$\text{Area of the wall} \approx \sum_k f(r(t_k)) \underbrace{|r(t_{k+1}) - r(t_k)|}_{\approx |r'(t_k)\Delta t|}$$

$$\approx \sum_k f(r(t_k)) |r'(t_k)| \Delta t$$

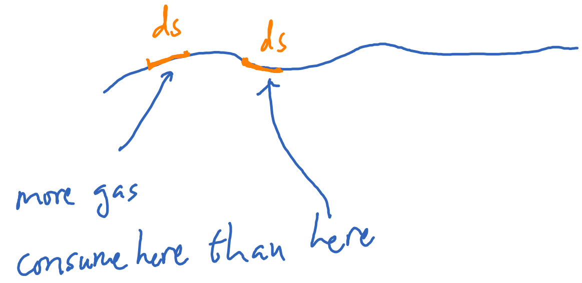
$$\text{Area of the wall} = \int_a^b f(r(t)) \underbrace{|r'(t)| dt}_{ds}$$

Def: The integral of a scalar function f along the curve C is

$$\int_C f \, ds = \int_a^b f(r(t)) |r'(t)| dt$$

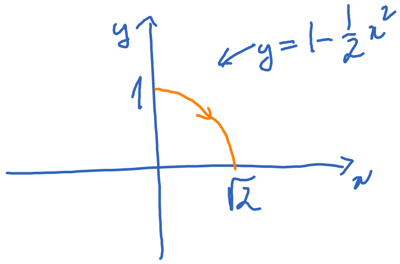
Interpretations:

- Area of a wall built on a curve
- Area under a curve carved on a surface
- Mass of a curve with uneven mass distribution
- Amount of gas consumed when driving on an uneven road.



$f(x,y)\,ds$ = amount of gas consumed by traveling a distance ds from the point (x,y) on the curve.

Ex



Find $\int_C xy\,ds$.

C: $r(t) = (t, 1 - \frac{1}{2}t^2)$, $0 \le t \le \sqrt{2}$.

$$ds = |r'(t)|\,dt = \sqrt{t^2+1}\,dt$$

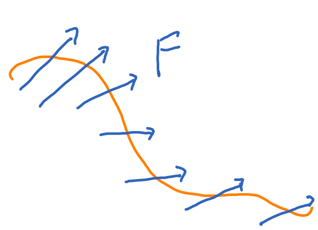
$$\int_C xy\,ds = \int_0^{\sqrt{2}} t\,t^2 \sqrt{\underbrace{t^2+1}_{u}}\,dt = \int_1^3 (u^2-1)\sqrt{u}\,\frac{1}{2}\,du$$

$(du = 2t\,dt)$

$$= \frac{1}{2}\int_1^3 \left(u^{5/2} - u^{1/2}\right) du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{2}{3}u^{3/2}\right)\Big|_1^3 = \cdots$$

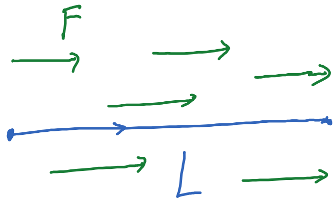
Line integral of a vector field



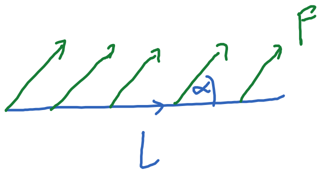
Work done by force field F along the curve:

$$\int_C F \cdot dr$$

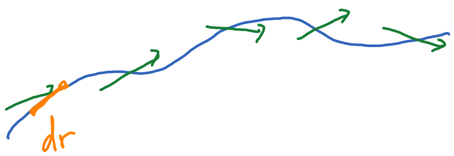
Why so?



work $= FL$



work $= FL\cos\alpha = \vec{F}\cdot\vec{L}$



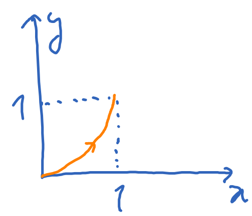
work $\approx \sum F\cdot\Delta r \approx \int_C F\cdot dr$

$\int_C F\cdot dr$ also represents the total effect of F on a particle moving on C: whether F pushes the particle along, or hinders the particle.

<u>Ex.</u>

$F(x,y) = (xy, -x^2)$

$r(t) = (t, t^2) \qquad 0 \le t \le 1$



$$\int_C F\cdot dr = \int_0^1 F(r(t))\cdot r'(t)\,dt$$

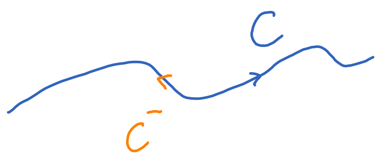
$$= \int_0^1 F(t,t^2)\cdot(1,2t)\,dt = \int_0^1 (tt^2, -t^2)\cdot(1,2t)\,dt$$

$$= \int_0^1 -t^3\,dt = -\frac{1}{4}$$

Another notation:

$$F = (P, Q), \quad dr = (dx, dy) \rightsquigarrow F \cdot dr = P\,dx + Q\,dy$$

Observation:



$C: \ r(t), \quad a \le t \le b$

$C^-: \ r(a+b-t) \quad a \le t \le b$

$$\int_C F \cdot dr = - \int_{C^-} F \cdot dr$$