Lecture 2

Friday, April 29, 2022 12:53 PM

* Prayer

* Spiritual thought : when we come to a new territory, we look for tools to work...

Calc I: $f(x)$, $x \in \mathbb{R}$

on $\mathbb{R}$, there are natural operations such as addition, multiplication, division, ... We are so used to these tools that we don't usually think about them.

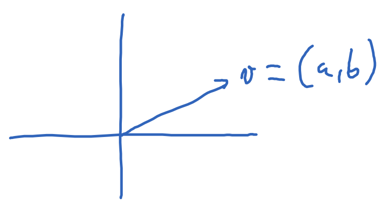
Multivariable Calc : $f(x,y)$ $(x,y) \in \mathbb{R}^2$

$f(x,y,z)$ $(x,y,z) \in \mathbb{R}^3$

Addition, multiplication, ... on $\mathbb{R}^2$ and $\mathbb{R}^3$?

In $\mathbb{R}^2$, $\mathbb{R}^3$, we also need a sense of direction (which is innate in $\mathbb{R}$).

The notion of vectors solve these problems.

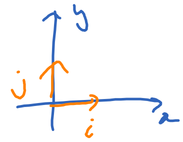


Note: textbook distinguishes (a,b) with $\langle a,b \rangle$. We don't.

In physics, vectors also has a specified starting point. For us, vectors are only specified by <u>length</u> and <u>direction</u>.

$v = (a,b) = a(1,0) + b(0,1)$

unit vectors i and j

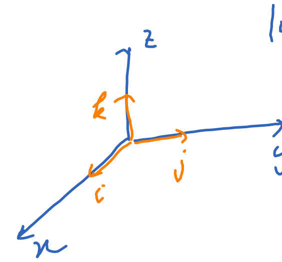


$|v| = \sqrt{a^2+b^2}$

$v = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

unit vectors i, j, k

$|v| = \sqrt{a^2+b^2+c^2}$



Ex

$v = (2,3,1)$

$w = (1,-1,2)$

what is $2v - 3w$?

Operations on vectors
- addition
- scaling
- dot product
- cross product
- triple product

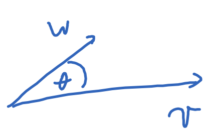
Dot product:

$$\left.\begin{array}{l} v = (a,b,c) \\ w = (d,e,f) \end{array}\right\} \quad v \cdot w = ad + be + cf$$

$\leftarrow$ This is an algebraic definition.

What is the geometric significance of dot product?

$\longrightarrow$ a means to compute the angle between two vectors

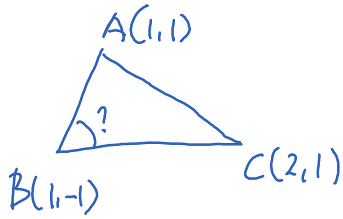


θ?

Theorem

$$\cos\theta = \frac{v \cdot w}{|v|\,|w|}$$

Ex



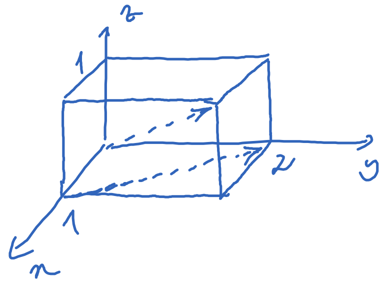
$\vec{BA} = (1-1,\ 1-(-1)) = (0,2)$

$\vec{BC} = (2-1,\ 1-(-1)) = (1,2)$

$$\cos B = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{0(1)+2(2)}{\sqrt{0^2+2^2}\sqrt{1^2+2^2}} = \frac{2}{\sqrt{5}}$$

$B \approx 27^\circ$

Ex



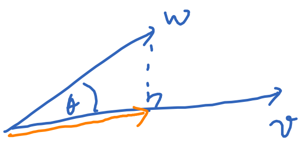
Dot product is used to check if the angle between two vectors is a right angle, obstuse, or acute angle.

$v.w = 0 \to$ right angle

$v.w < 0 \to$ obstuse

$v.w > 0 \to$ acute

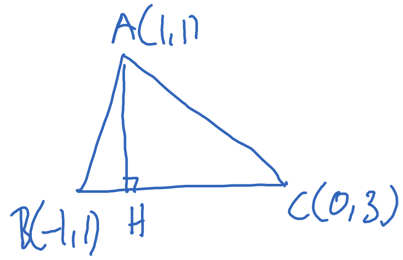
Projection:



$$\text{Comp}_v w = |w|\cos\theta = |w|\frac{v.w}{|v||w|} = \frac{v.w}{|v|}$$

$$\text{proj}_v w = (\text{Comp}_v w)\underbrace{\frac{v}{|v|}}_{\text{unit vector}} = \frac{v.w}{|v|^2}\, v$$

Ex



What is $\overrightarrow{AH}$?

$\overrightarrow{BA} = (2,0) = w$

$\overrightarrow{BC} = (1,2) = v$

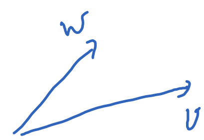
$$\overrightarrow{BH} = \text{proj}_{\overrightarrow{BC}} \overrightarrow{BA} = \text{proj}_v w = \frac{v \cdot w}{|v|^2} v = \frac{2(1)+0(2)}{\sqrt{1^2+2^2}^{\,2}} (1,2) = \frac{2}{5}(1,2)$$

$$= \left(\frac{2}{5}, \frac{4}{5}\right)$$

Cross product: defined for vectors in $\mathbb{R}^3$ only

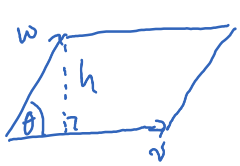
algebraic

geometric



$v \times w$ is a vector:

- perpendicular to both v and w
- directed according to the right hand rule
- length = area of the parallelogram formed by v and w.



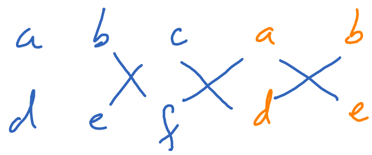
$$\text{area} = h\,|v| = |w| \sin\theta \, |v|$$

$$= |v||w| \sin\theta$$

It is easier to compute cross product algebraically.

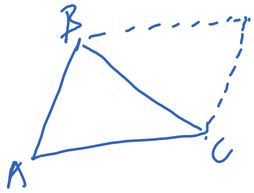
$v = (a,b,c)$

$w = (d,e,f)$



$v \times v = (bf - ce,\ cd - af,\ ae - bd)$

Ex Find the area of the triangle formed by $A(1,2,3)$, $B(0,1,-1)$, $C(2,-1,-1)$.



$$\text{area} = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

Ex What is a normal vector to the plane formed by A, B, C?

Triple integral $a \cdot (b \times c)$, used to check if a, b, c are coplanar.

a, b, c are coplanar iff $a \cdot (b \times c) = 0$.