Lecture 5

Thursday, May 5, 2022 10:45 PM

* Prayer

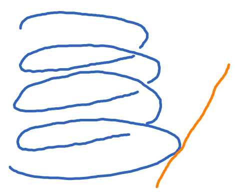
* Spiritual thought

* Revisit an example last time:

$$r(t) = (\cos t, \sin t, t)$$

Find the tangent line to the curve at point $(1,0,0)$.

Point $(1,0,0)$ corresponds to $t=0$. A tangent vector to the curve at this point is $r'(0)$.



$$r'(t) = (-\sin t, \cos t, 1)$$

$r'(0) = (0,1,1)$ – this is also a direction vector to the tangent line.

The tangent line also passes through $(1,0,0)$.

Parametric eq. $\begin{cases} x = 1 \\ y = t \\ z = t \end{cases}$

* Integral:

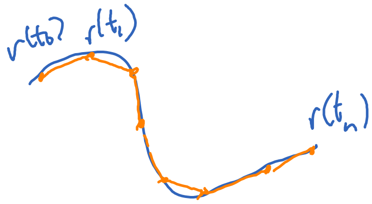
$$\int r(t)\,dt = \left(\int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt\right)$$

We will revisit it when we talk about velocity.

Geometric properties of curves

These are properties that don't depend on the parametrizations of the curve.

They include $\begin{cases} \text{length} \\ \text{curvature} \\ \text{torsion} \end{cases}$



Length: $r(t)$, $a \le t \le b$

$$L \approx |r(t_1) - r(t_0)| + \dots + |r(t_n) - r(t_{n-1})| = \sum |r(t_{j+1}) - r(t_j)|$$

$$\frac{r(t_{j+1}) - r(t_j)}{\Delta t} \approx r'(t_j)$$

$$\leadsto \quad |r(t_{j+1}) - r(t_j)| \approx |r'(t_j)| \Delta t$$

$$L \approx \sum_j |r'(t_j)| \Delta t \xrightarrow{n \to \infty} \int_a^b |r'(t)|\, dt = L$$

Ex find the length of the curve parametrized by

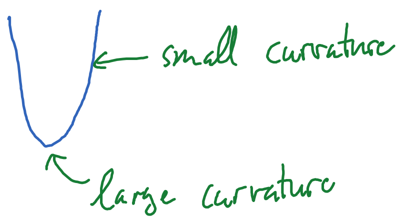
$$\begin{cases} x = t \\ y = \frac{4}{3} t^{3/2} \\ z = t^2 \end{cases} \qquad 0 \le t \le 1$$

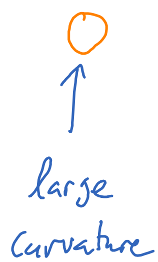
$$r(t) = \left(t, \tfrac{4}{3} t^{3/2}, t^2\right) \leadsto r'(t) = \left(1, 2t^{1/2}, 2t\right)$$

$$|r'(t)| = \sqrt{1+4t+4t^2} = \sqrt{(1+2t)^2} = 1+2t.$$

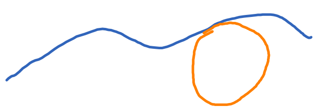
$$L = \int_0^1 |r'(t)|\,dt = \int_0^1 (1+2t)\,dt = (t+t^2)\Big|_0^1 = 2$$

Curvature

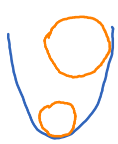




Curvature of a curve at a point $= \frac{1}{\text{radius}}$ of the circle that best fits the curve at that point

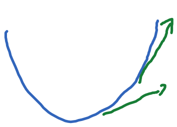


Curvature of a circle is equal to $\frac{1}{R}$.



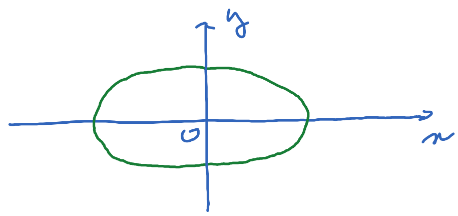
How to calculate the curvature from a parametrization?

$$\kappa = \left|\frac{dT}{ds}\right| \quad \text{, where } T \text{ is a unit tangent vector}$$



$$= \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

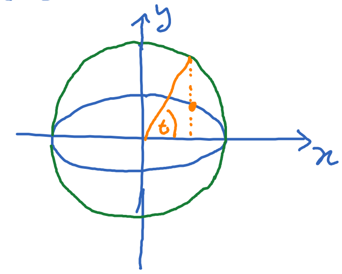
Ex



Find the points on the ellipse $x^2+4y^2=9$ that has minimum/maximum curvature.

Parametrize the curve:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{2y}{3}\right)^2 = 1 \rightsquigarrow \begin{cases} x = 3\cos t \\ y = \frac{3}{2}\sin t \\ z = 0 \end{cases} \qquad 0 \le t \le 2\pi$$



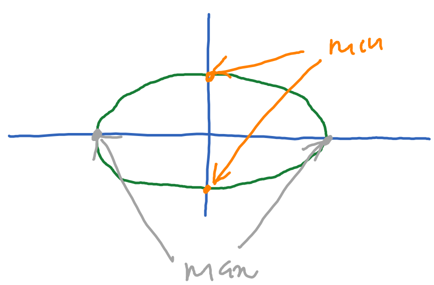
$$r(t) = \left(3\cos t, \frac{3}{2}\sin t, 0\right)$$

$$\left.\begin{aligned} r'(t) &= \left(-3\sin t, \frac{3}{2}\cos t, 0\right) \\ r''(t) &= \left(-3\cos t, -\frac{3}{2}\sin t, 0\right) \end{aligned}\right\} r' \times r'' = \left(0, 0, \frac{9}{2}\sin^2 t + \frac{9}{2}\cos^2 t\right) = \left(0, 0, \frac{9}{2}\right)$$

$$k = \frac{|r' \times r''|}{|r'|^3} = \frac{9/2}{\left(9\sin^2 t + \frac{9}{4}\cos^2 t\right)^{3/2}} = \frac{9/2}{\left(\frac{9}{4} + \frac{27}{4}\sin^2 t\right)^{3/2}}$$

k is max when $\sin t = 0 \rightsquigarrow t = 0$ or $t = \pi$.

k is min when $\sin t = \pm 1 \rightsquigarrow t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$.



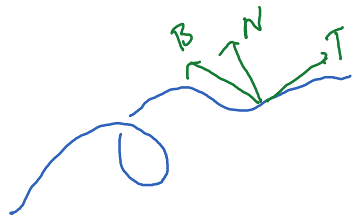
Torsion

Torsion is a measure of how much a curve twists.

A curve in 2D can bend but doesn't twist

A curve in 3D can bend and twist.

Imagine the roller coaster.



N : normal vector

B : binormal vector

τ = how fast B changes its direction

κ = how fast T changes its direction

Formula:

$$\tau = -\frac{dB}{ds} \cdot N = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}$$

If $\tau > 0$ then the curve is moving like a spiral with positive orientation.

If $\tau < 0$ " " " " " negative "

Ex

$r(t) = (\cos t, \sin t, at)$ where a is a constant

Find the torsion (depending on a). For which value of a is the torsion maximum?

$$\left.\begin{array}{l} r' = (-\sin t, \cos t, a) \\ r'' = (-\cos t, -\sin t, 0) \end{array}\right\} \; r' \times r'' = (a \sin t, -a \cos t, 1)$$

$$r''' = (\sin t, -\cos t, 0) \longrightarrow (r' \times r'') \cdot r''' = a \sin^2 t + a \cos^2 t = a$$

$$\tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2} = \frac{a}{a^2 + 1}$$