Lecture 7

Wednesday, May 11, 2022 11:43 AM

* Prayer

* Spiritual thought

* Limits of a multivariable function:

$$\lim_{(x,y)\to(a,b)} f(x,y) \qquad \lim_{\substack{x\to a\\ y\to b}} f(x,y)$$

x tends to a, and y tends to b independently of each other.

To show that the limit doesn't exist, we can either

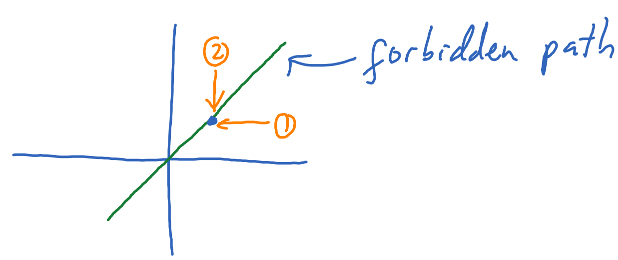
- find two paths leading to (a,b) that yield two different limits,

or

- find one path leading to (a,b) where the limit doesn't exist.

Ex

$$\lim_{(x,y)\to(1,1)} \frac{x+y}{x-y}$$



On path 1: $x=t,\ y=0,\ t\to 1$

$$\frac{x+y}{x-y} = \frac{t+0}{t-0} = 1 \longrightarrow 1$$

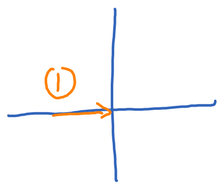
On path 2: $x=0,\ y=t,\ t\to 1$

$$\frac{x+y}{x-y} = \frac{0+t}{0-t} = -1 \longrightarrow -1$$

Two paths yield two different limits. Thus, $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x+y}{x-y}$ doesn't exist.

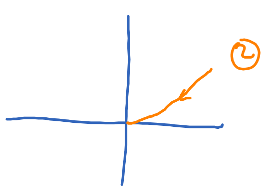
Ex:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2-y^2}$$



On path ①: $\dfrac{x^3}{x^2-y^2} = \dfrac{x^3}{x^2-0} = x \longrightarrow 0$

Note that the numerator and denominator can approach 0 at two different rates (depending on how close x and y are to each other).



On path ②: $y = x + x^2$

$$\frac{x^3}{x^2-y^2} = \frac{x^3}{x^2-(x+x^2)^2} = \frac{x^3}{x^2-(x^2+2x^3+x^4)} = \frac{x^3}{-2x^3-x^4} \to -\frac{1}{2}$$

Thus, the limit doesn't exist.

* To show the limit exists, we try to eliminate the indefinite form or try to use squeeze theorem.

Ex

$$\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2-y^2} = \lim \frac{(x-y)}{(x-y)(x+y)} = \lim \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$$

Ex

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

$$-|y| \le \frac{x^2 y}{x^2+y^2} \le |y|$$

$$\searrow \quad 0 \quad \swarrow$$

Thus, the limit is equal to 0 by squeeze theorem.

Food for thought:

Is $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$ when both limits exist?

If yes, explain why. If no, give a counter example.

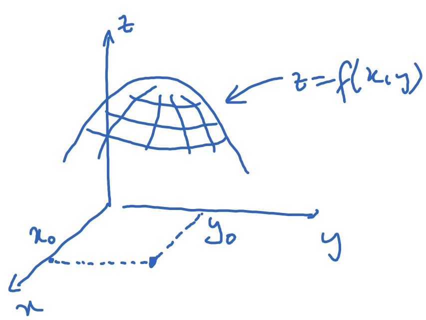
3 plus points for any person submitting a correct answer on Friday.

Derivative

$f(x,y)$ What would be $f'(x,y)$?

Calc I: $\displaystyle f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ = slope of tangent line

= instantaneous rate of change



f has two variables. These variables change in values causing f to change its value. The simplest way to see how each variable affects the value of f is by freezing one variable at a time.

Freeze y: $f(x,y)$ has only one variable x

The rate of change wrt x is

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$

which is denoted by $\frac{\partial f}{\partial x}$ (partial derivative wrt x)

Similarly, the rate of change wrt y is

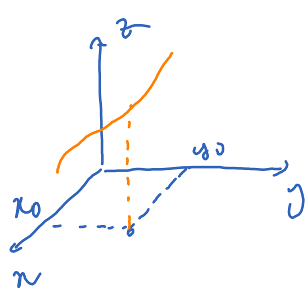
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Ex ① $f(x,y) = x^2 + 2xy^2 + y^3$

② $f(x,y) = x^y$

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

Geometric meaning of partial derivatives

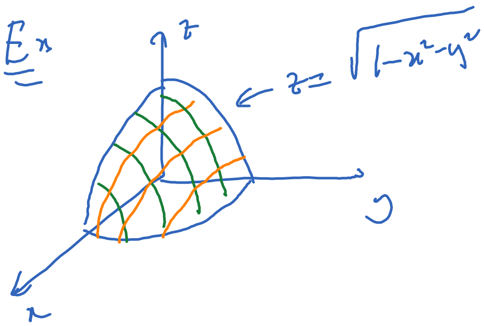


$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

$\frac{\partial f}{\partial x}(x_0, y_0)$ = slope of the tangent line to the curve at the intersection of the plane $y = y_0$ and the surface $z = f(x, y)$.

Ex



$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}} < 0$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}} < 0$$