MATH 314, MIDTERM, SPRING 2022

INSTRUCTOR: TUAN PHAM

Name

Instructions:

- This is a closed-book exam, 2 hours and 30 minutes long. No calculators or notes are allowed.
- For Problems 1-15, fill in the bubbles on this front page. To each problem, only one answer is correct.
- For Problems 16, 17, 18, make sure to show all necessary steps. Mysterious answers will receive little or no credit.
- Some formulae are provided below.
- Do not discuss the exam with anyone during May 19-21.

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14. Ⓐ Ⓑ **Ⓒ** Ⓓ
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Problem	Possible points	Earned points
1-15	30	
16	10	
17	10	
18	10	
Total	60	

$$\kappa = \frac{|r' \times r''|}{|r'|^3}, \quad \tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}, \quad N = \frac{T'}{|T'|}, \quad a_T = \frac{r' \cdot r''}{|r'|}, \quad a_N = \frac{|r' \times r''|}{|r'|}.$$

Problem 1. (2 points) Consider three points $A(1,2,3)$, $B(2,3,4)$, $C(3,2,3)$. Which of the following statements is true ?

A. $|\overrightarrow{AB}| = |\overrightarrow{AC}|$

B. $|\overrightarrow{AB}| = |\overrightarrow{BC}|$

C. $|\overrightarrow{BC}| = |\overrightarrow{AC}|$

D. None of the above

Problem 2. (2 points) The set of triples (x, y, z) satisfying $x^2 + 2x + y^2 - 2y + z^2 = 2$ is a sphere

A. centered at $(-1, 1, 0)$ with radius 2

B. centered at $(1, -1, 0)$ with radius 2

C. centered at $(-1, 1, 0)$ with radius $\sqrt{2}$

D. centered at $(1, -1, 0)$ with radius $\sqrt{2}$

Problem 3. (2 points) For what value of x is the angle between vector $(1, x, 1)$ and vector $(0, 1, 1)$ is $45°$?

A. -1

B. $-1/2$

C. 0

D. $1/2$

Problem 4. (2 points) Find the area of the triangle with vertices at $A(3,1,1)$, $B(2,1,0)$, $C(-1,2,2)$.

A. $\frac{\sqrt{27}}{2}$

B. $\frac{\sqrt{19}}{2}$

C. $\frac{\sqrt{11}}{2}$

D. $\frac{\sqrt{10}}{2}$

Problem 5. (2 points) The surface given by $x = y^2 + 3z^2$ is called a/an

A. ellipsoid
B. cone
C. elliptic paraboloid
D. hyperbolic paraboloid

Problem 6. (2 points) At what points does the line parametrized by $x = t$, $y = 0$, $z = t - 1$ intersect the sphere $x^2 + y^2 + z^2 = 1$?

A. $(0, 0, -1)$ and $(-1, 0, 0)$
B. $(0, 0, -1)$ and $(1, 0, 0)$
C. $(0, 0, 1)$ and $(-1, 0, -2)$
D. $(0, 0, 1)$ and $(1, 0, 0)$

Problem 7. (2 points) The line tangent to the curve $r(t) = (t^2 + t,\ 4e^{t-1},\ t + 1)$ at the point $(2, 4, 2)$ has parametric equations

A. $x = 3 + 4t,\ y = 4 + 4t,\ z = 1 + 2t$

B. $x = 2t^2 + t + 2,\ y = 4te^{t-1} + 4,\ z = t + 2$

C. $x = 2 + 3t,\ y = 4 + 4t,\ z = 2 + t$

D. $x = t^3 + t^2 + 3,\ y = 4te^{t-1} + 4,\ z = t^2 + t + 1$

Problem 8. (2 points) The length of the spiral curve $r(t) = (\sin t,\ t,\ \cos t)$, where $0 \leq t \leq \pi$ is

A. 2π
B. $\pi/2$
C. π^2
D. $\pi\sqrt{2}$

Problem 9. (2 points) A vector function $r(t)$ satisfies $r'(t) = (t, t^2)$ and $r(1) = (1, 1)$. Which of the following is the correct formula of $r(t)$?

A. $\left(\frac{t^2}{2} + 1,\ \frac{t^3}{3} + 1\right)$

B. $\left(\frac{t^2+1}{2},\ \frac{t^3+2}{3}\right)$

C. $(1,\ t)$

D. $(t,\ 1)$

Problem 10. (2 points) Which of the following statements is false about the motion of an object along the unit circle?

A. The velocity is always tangent to the circle.
B. The acceleration is always equal to the derivative of the velocity.
C. The acceleration always points toward the inside of the circle.
D. The tangential acceleration is always equal to zero.

Problem 11. (2 points) The position function of a moving particle is given by $r(t) = (-\frac{t^2}{2}, t)$. What is the speed of the particle at time $t = 1$?

A. $\frac{1}{\sqrt{2}}$

B. $\sqrt{2}$

C. $\frac{1}{2\sqrt{2}}$

D. $(-1, 1)$

Problem 12. (2 points) A contour map of the function $f(x, y) = \sqrt{1 - 9x^2 - 4y^2}$ consists of

A. ellipses
B. parabolas
C. hyperbolas
D. concentric circles

Problem 13. (2 points) Find $\lim_{(x,y)\to(0,0)} \dfrac{x^3 - y^2}{x^2 + y^2}$.

A. 1
B. -1
C. 0
D. does not exist

Problem 14. (2 points) The plane tangent to the graph of $f(x, y) = x^2 + xy + y^2$ at the point $(1, -1, 1)$ has an equation

A. $z = x - y + 3$
B. $z = -x + y + 3$
C. $z = x - y - 1$
D. $z = -x + y - 1$

Problem 15. (2 points) Let $f(x, y) = \ln(x^2+y^2)$. Which of the following is the correct formula of f_{xx} ?

A. $\dfrac{-4xy}{x^2 + y^2}$

B. $\dfrac{4xy}{x^2 + y^2}$

C. $\dfrac{2(y^2 - x^2)}{x^2 + y^2}$

D. $\dfrac{2(x^2 - y^2)}{x^2 + y^2}$

Problem 16. (10 points) Write the equation of the plane passing through three points $\underbrace{(2,-1,1)}_{A}$, $\underbrace{(1,0,2)}_{B}$, $\underbrace{(1,1,4)}_{C}$.

$$\left.\begin{array}{l}\vec{AB} = (-1, 1, 1) \\ \vec{BC} = (0, 1, 2)\end{array}\right\} \quad \vec{AB} \times \vec{BC} = (1, 2, -1)$$ is a normal vector of the plane.

Eq. of the plane:

$$1(x-2) + 2(y+1) + (-1)(z-1) = 0$$

or

$$x + 2y - z = -1$$

Problem 17. (10 points) At what point does the curve $y = e^x$ (on the xy-plane) have a maximum curvature? Find that curvature.

$$r(t) = (t, e^t, 0)$$

$$\left.\begin{aligned} r'(t) &= (1, e^t, 0) \\ r''(t) &= (0, e^t, 0) \end{aligned}\right\} r' \times r'' = (0, 0, e^t)$$

$$\kappa = \frac{|r' \times r''|}{|r'|^3} = \frac{e^t}{(1+e^{2t})^{3/2}}$$ ← want to maximize this function.

Let $u = e^t > 0$.

$$\kappa = \frac{e^t}{(1+e^{2t})^{3/2}} = \frac{u}{(1+u^2)^{3/2}} = f(u)$$

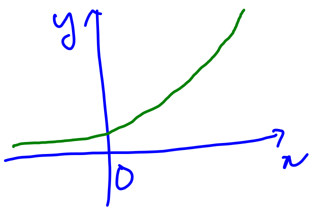
Find critical points of f:

$$f' = \frac{(1+u^2)^{3/2} - \frac{3}{2}2u\,u\,(1+u^2)^{1/2}}{(1+u^2)^3} =$$

$$= \frac{(1+u^2)^{3/2} - 3u^2(1+u^2)^{1/2}}{(1+u^2)^3} = \frac{(1+u^2)^{1/2}\left((1+u^2) - 3u^2\right)}{(1+u^2)^3}$$

$$= \frac{1-2u^2}{(1+u^2)^{5/2}}$$

$$f' = 0 \rightsquigarrow 1 - 2u^2 = 0 \rightsquigarrow u = \frac{1}{\sqrt{2}} \quad \text{(because } u > 0\text{)}.$$



$$e^t = u = \frac{1}{\sqrt{2}} \rightsquigarrow t = \ln\left(\frac{1}{\sqrt{2}}\right)$$

: this is where κ is maximum.

$$\kappa_{max} = \max f(u) = f\left(\frac{1}{\sqrt{2}}\right) = \frac{1/\sqrt{2}}{\left(1+\frac{1}{2}\right)^{3/2}} = \frac{1}{\sqrt{2}} \frac{2^{3/2}}{3^{3/2}} = \frac{2}{3^{3/2}}.$$

Problem 18. (10 points) Let $w = xy^2 + yz + z^2$, where $x = 2u - v$, $y = u + v$, and $z = uv$. View w as a function of u and v. *Use the chain rule* to find $w_u(2, 1)$.

By chain rule,

$$w_u = w_x x_u + w_y y_u + w_z z_u$$

$$\begin{cases} w_x = y^2 \\ w_y = 2xy + z \\ w_z = y + 2z \end{cases} \qquad \begin{cases} x_u = 2 \\ y_u = 1 \\ z_v = v \end{cases}$$

$$\rightsquigarrow \quad w_u = 2y^2 + (2xy + z) + v(y + 2z).$$

When $u = 2$ and $v = 1$, we get $x = 3$, $y = 3$, $z = 2$.

Therefore,

$$w_u(2,1) = 2(3)^2 + (2(3)(3) + 2) + 1(3 + 2(2))$$

$$= 18 + 20 + 7$$

$$= 45$$