Lecture 11

Thursday, February 18, 2021 2:24 PM

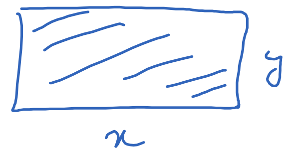
* Prayer
* Spiritual thought
* Answering questions

Chain rule

$$z = f(x,y) = f(x(t), y(t))$$

Ex

$$z = f(x,y) = xy$$



$x = t^2$

$y = t$

what is $\frac{dz}{dt}$?

$\Delta z = x\Delta y + y \Delta x$

$z' = xy' + yx'$

Recall : $\left[f(g(x))\right]' = \underbrace{f'(g(x))}_{\text{der. of the "outer" function}} \underbrace{g'(x)}_{\text{der. of the "inner" function}}$

$$\Delta f(g) \approx f'(g)\,\Delta g \approx f'(g)\, g' \Delta x.$$

Now $f = f(x,y)$

$$\Delta f \approx f_x \Delta x + f_y \Delta y \approx f_x\, x' \Delta t + f_y\, y' \Delta t$$

$$\rightsquigarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$f(x,y) = f(x(u,v,w),\, y(u,v,w))$$

$$df = f_x\,dx + f_y\,dy = f_x \cdot (x_u\,du + x_v\,dv + x_w\,dw)$$

$$+ f_y \cdot (y_u\,du + y_v\,dv + y_w\,dw)$$

$$= (f_x x_u + f_y y_u)\,du + (f_x x_v + f_y y_v)\,dv + (f_x x_w + f_y y_w)\,dw$$

Thus,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \cdots\cdot$$

$$\frac{\partial f}{\partial w} = \cdots\cdot$$

Ex:

$z = (x+y)^4$

$x = s^2 t$ Find $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial s}$.

$y = t+s$

Ex:

$z = \ln\sqrt{x^2+y^2+z^2}$

$x = t^2$ Find $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial s}$.

$y = ts$

$z = s^2$

* Implicit differentiation

$$x^2 + yz + z^2 x = 1 \qquad (*)$$

$$x = 1,\ y = -2$$

$$\rightsquigarrow z^2 - 2z = 0$$

$$\longrightarrow z = 0 \text{ or } z = 2$$

$$z = z(x, y)$$

Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ at $(1, -2)$ (with $z = 0$)

Differentiating $(*)$ w.r.t x:

$$2\underset{\substack{\downarrow \\ 1}}{x} + \underset{\substack{\downarrow \\ -2}}{y} z_x + 2\underset{\substack{\downarrow \\ 0}}{z} z_x \underset{\substack{\downarrow \\ 1}}{x} + \underset{\substack{\downarrow \\ 0}}{z^2} = 0 \longrightarrow z_x = 1.$$

Ex

$$y \sin x = x^2 - y^2 + 1$$

$$\left.\frac{dy}{dx}\right|_{x=0,\, y=1} = ?$$

* Reminder of the chain rule:

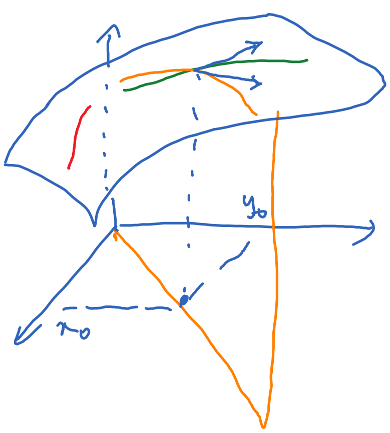
$$f(x,y), \quad x = u^2+v^2+w^2, \quad y = uvw$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \dots$$

Ex:
$$g(u,v) = f(u^2+v^2, uv)$$

$$\frac{\partial g}{\partial u} = ?, \quad \frac{\partial g}{\partial v} = ?$$

* Directional derivatives:



$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

$$u = \langle a,b \rangle$$

$$\underbrace{\frac{f(x_0+ah, y_0+bh) - f(x_0, y_0+bh)}{h}}_{\to\, a f_x(x_0,y_0)} + \underbrace{\frac{f(x_0, y_0+bh) - f(x_0,y_0)}{h}}_{\to\, b f_y(x_0,y_0)}$$

Ex $f(x,y) = 2xy^2$ at point $(x_0, y_0) = (2,1)$

① Find the rate of change of f at (x_0, y_0) in the direction of vector $\langle 3,2 \rangle$.

② What is the direction that gives the maximum rate of change?

* Tangent plane to a surface

Gradient vector is always perpendicular to the level sets.