Lecture 19

Thursday, March 18, 2021 2:00 PM

* Prayer
* Spiritual thought
* Answering questions....

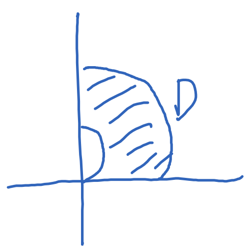
Change of variables

$$\iint_D f(x,y)\,\underbrace{dx\,dy}_{dA} = \iint_{D'} f(x(u,v), y(u,v)) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}_{dA}$$

Ex

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

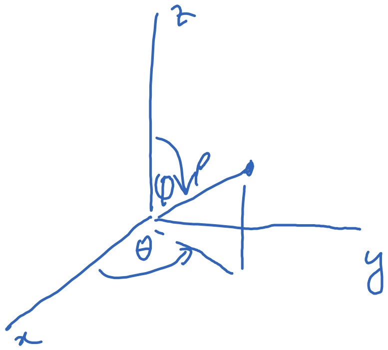
$$\frac{dx\,dy}{du\,dv} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$



$$\iint_D (r+\theta)\,dA = \iint_{D'} \left(\sqrt{x^2+y^2} + \arctan\left(\frac{y}{x}\right)\right) dx\,dy$$

$$D = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2},\ \theta \le r \le \theta + 1\}$$

Spherical coords



$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases} \qquad (x, y, z) \longmapsto (\rho, \theta, \phi)$$

* Convert $(1, -\sqrt{3}, 2\sqrt{3})$ into spherical coords.

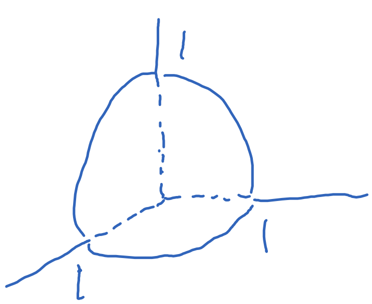
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{16} = 4$$

$$\cos\phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \rightsquigarrow \phi = \frac{\pi}{6}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{\rho\sin\phi} = \frac{1}{4(\frac{1}{2})} = \frac{1}{2} \\ \sin\theta &= \frac{y}{\rho\sin\theta} < 0 \end{aligned}\right\} \theta = \frac{2\pi}{3}$$

So $(\rho, \theta, \phi) = \left(4, \frac{2\pi}{3}, \frac{\pi}{6}\right)$.

* Triple integral:



$$\iiint_E x \underbrace{dV}_{dxdydz} = \iiint_{E'} \rho\sin\phi\cos\theta\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3 \sin^2\phi \cos\theta \, d\rho\, d\theta\, d\phi = \cdots$$

Vector fields

$f = x^3 + xy^2 + y^2 \rightsquigarrow$ potential function

$F = \langle 3x^2 + y^2, 2xy + 2y \rangle \rightsquigarrow$ conservative vector field.

$f_x = 3x^2 + y^2 \rightsquigarrow f(x,y) = x^3 + xy^2 + C(y)$

$f_y = 2xy + C'(y)$

$\rightsquigarrow C'(y) = 2y$

$\rightsquigarrow C(y) = y^2$

Ex (non-conservative vector field)

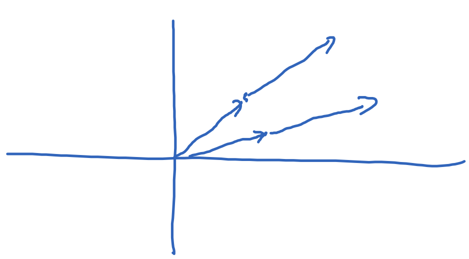
$F(x,y) = \langle y^2, x^2 \rangle$

$y^2 = f_x \rightsquigarrow f = xy^2 + C(y)$

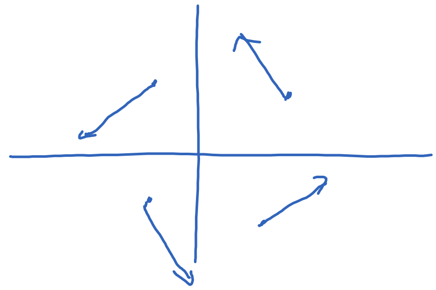
$f_y = 2xy + C'(y) = x^2$

$\rightsquigarrow$ there is no such f!

Plot vector field ....

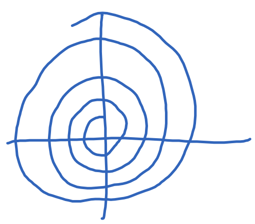


$f(x,y) = \langle x, y \rangle$



$f(x,y) = \langle y, x \rangle$

Examples of vector fields: gravity, velocity, magnetic field, gradient vectors, force, ...



$r = f(\theta) = e^{\theta} \sin\theta$

$\theta \to \infty$: $r \to \infty$

$\theta \to -\infty$: $r \to 0$

Plot vector fields on Mathematica:

VectorPlot[{x^2, xy}, {x,0,1}, {y,0,1}]

Line integral

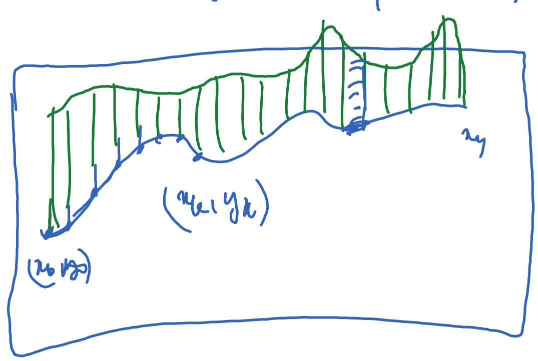
integral — "Single" integral $\int_a^b$, $\int_{[a,b]}$

— double integral $\iint_D$

— triple integral $\iiint_E$

— iterated integral $\int_a^b \int_c^d \cdots$

Can we integrate a function $f(x,y)$ over a curve instead of a region?



$$s(t) = \int_a^t |r'(\tau)|\, d\tau$$

$$ds = s'\,dt = |r'(t)|\,dt$$

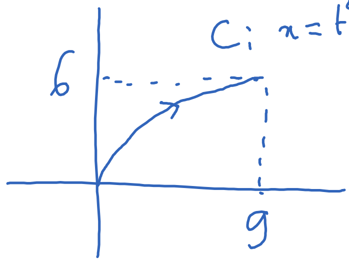
$$\sum f(x_k, y_k)\,|r'(t_k)|\,\Delta t$$

$$\approx \int_a^b f(x(t), y(t))\, \underbrace{|r'(t)|\,dt}_{ds}$$

$$= \text{"}\int_C f(x,y)\, ds\text{"}$$

integral of a scalar function over a curve.

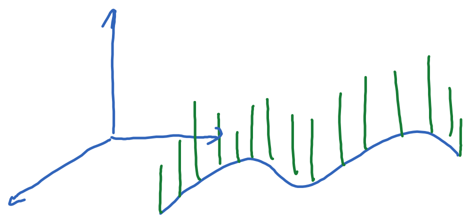
Ex



$$\int_C y\,ds = ?$$

Mathematica:

how to plot function $f(x,y)$ on top of a curve C?



$C: \ (x(t), y(t), 0)$

Surface: $(x(t), y(t), s)$

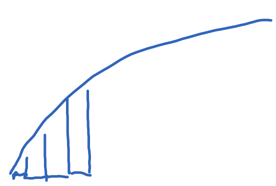
$t \in [a,b], \ s \in [0, f(x(t), y(t))]$

ParametricPlot3D[{t^2, 2t, s}, {t, 0, 3}, {s, 0, 2t}]

$\int_C f(x,y)\,ds$ is the area of the wall.

or mass of a wire

* <u>Line integral w.r.t x or y</u>



$$\int_C f(x,y)\,dx$$