Lecture 22

Tuesday, March 30, 2021 3:48 PM

* Prayer
* Spiritual thought
* Answering questions ...

Remark about orientation of a curve:

C: $r(t)$ $\quad a \le t \le b$

C^-: $\tilde{r}(t) = r(a+b-t)$

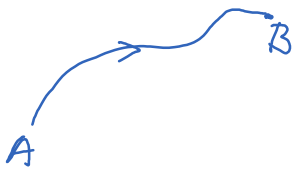
$$\int_C f(x,y)\,ds = \int_{C^-} f(x,y)\,ds$$

$$\int_C f(x,y)\,dx = -\int_{C^-} f(x,y)\,dx$$

$$\int_C F \cdot dr = -\int_{C^-} F \cdot dr$$

Fundamental theorem of calculus:

$$\int_C \underbrace{\nabla f}_{\text{conservative}} \cdot dr = f(B) - f(A)$$



Note: $\int_C f_x(x,y)\, ds \neq f(B) - f(A)$

$\int_C f_x(x,y)\, dx \neq f(B) - f(A)$

How to check if a vector field is conservative?

$F = \langle P, Q \rangle$

Method 1: $F = \langle 2xy, x^2 \rangle$

$2xy = f_x$, $x^2 = f_y$

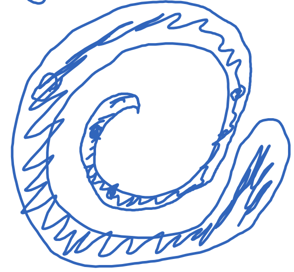
$f(x,y) = x^2 y + C(y)$

$f_y = x^2 + C'(y) \rightsquigarrow C(y) = 0$

Method 2 $F = \langle P, Q \rangle$

$P = f_x$, $Q = f_y$

- If $P_y \neq Q_x$ then F is not conservative.
- If $P_y = Q_x$ AND D is simply connected then F is conservative.
- If $P_y = Q_y$ and D is not simply connected then we can't conclude.

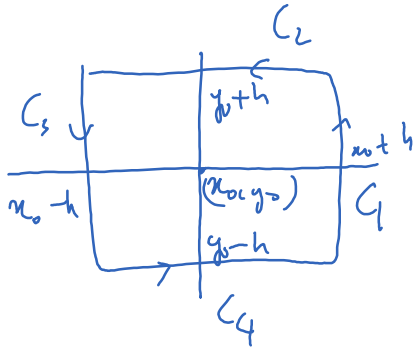


Ex

$F(x,y) = \langle xy + y^2,\ x^2 + 2xy \rangle$

$F(x,y) = \langle y^2 - 2x,\ 2xy \rangle$

Green's theorem



$C_1: \begin{cases} x = x_0 + h \\ y = y_0 + t \end{cases} \quad -h \le t \le h$

$C_2: \begin{cases} x = x_0 - t \\ y = y_0 + h \end{cases} \quad -h \le t \le h$

$C_3: \begin{cases} x = x_0 - h \\ y = y_0 - t \end{cases} \quad -h \le t \le h$

$C_4: \begin{cases} x = x_0 + t \\ y = y_0 - h \end{cases} \quad -h \le t \le h$

Circulation $= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy$

($\int_{C_1} Q\,dy = 0$, $\int_{C_2} Q\,dy = 0$, $\int_{C_3} Q\,dy = 0$, $\int_{C_4} Q\,dy = 0$)

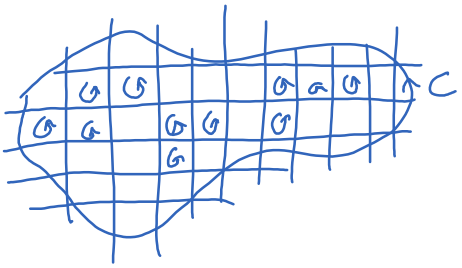
$= \int_{C_1} Q\,dy + \int_{C_2} P\,dx + \int_{C_3} Q\,dy + \int_{C_4} P\,dx$

$= \int_{-h}^{h} \left[Q(x_0+h, y_0+t) - Q(x_0-h, y_0+t) \right] dt - \int_{-h}^{h} \left[P(x_0+t, y_0+h) - P(x_0-t, y_0+h) \right] dt$

$\approx 4h^2 Q_x(x_0, y_0) - 4h^2 P_y(x_0, y_0)$

Circulation density $= \dfrac{\text{total circulation}}{\text{area} = \Delta A}$

$= Q_x(x_0, y_0) - P_y(x_0, y_0)$.



Total circulation

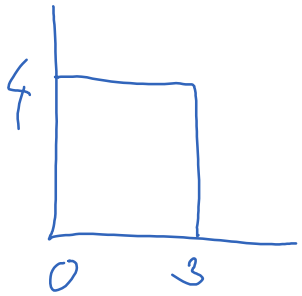
$= \int_C F \cdot dr$

Total circulation $= \iint_D (Q_x - P_y)\, dA$

Green's theorem:

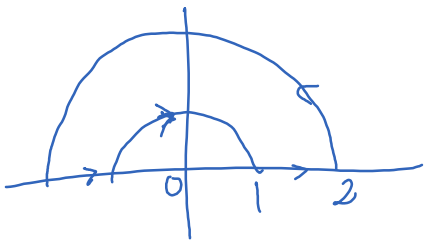
$$\int_C F \cdot dr = \iint_D (Q_x - P_x)\, dA$$

Ex



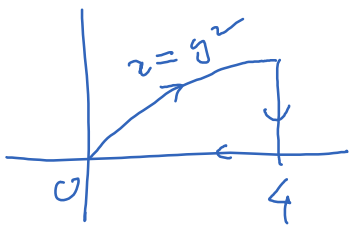
$$\int_C ye^x\, dx + 2e^x\, dy$$

Ex



$$\int_C y^2\, dx + 3xy\, dy$$

Ex



$$\int_C (x^{2/3} + y^2)\, dx + (y^{4/3} - x^2)\, dy$$