Lecture 5

Tuesday, January 26, 2021 2:46 PM

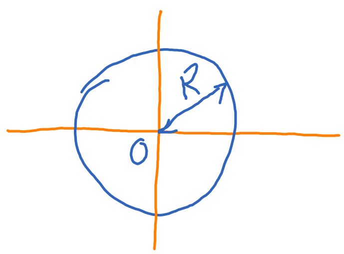
* Prayer

* Spiritual thought: lift where you stand

→ heaven will shake for your good

* Answering questions ....

A curve (as we refer to in this course) is more than just the shape of the curve. It should tell us how to draw it.



A circle can be parametrized as

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad 0 \le t \le 2\pi$$

or

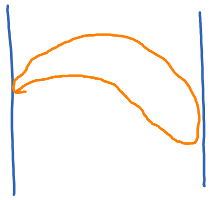
$$\begin{cases} x = R\sin t \\ y = R\cos t \end{cases} \qquad 0 \le t \le 2\pi$$

or ....

Each parametrization is considered a curve.

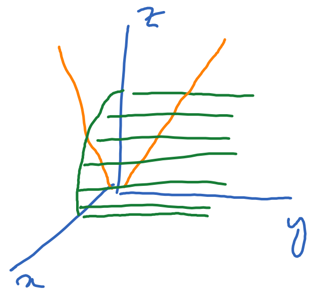
Ex Find a parametrization of the intersection of two surfaces

$$(S_1): x^2 + y^2 = 4 \quad , \qquad (S_2): z = xy$$



Ex Intersection of (S_3): $z = \sqrt{x^2+y^2}$

(S_4): $x^2+z^2 = 4 \rightsquigarrow z = \sqrt{4-x^2}$



$y = t$

$z^2 - x^2 = y^2$

$z^2 + x^2 = 4$

$\longrightarrow z^2 = \dfrac{y^2+4}{2} \rightsquigarrow z = \sqrt{\dfrac{t^2+4}{2}}$

$x = z\cos t$

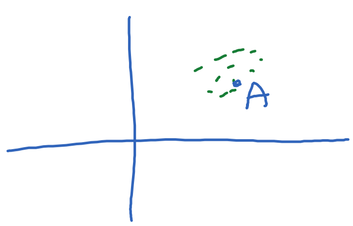
$4 - x^2 = x^2 + y^2 \rightsquigarrow 2x^2 + y^2 = 4$

$\rightsquigarrow x = \sqrt{2}\cos t, \quad y = 2\sin t$

$z = \sqrt{x^2+y^2} = \sqrt{2\cos^2 t + 4\sin^2 t}$

Vector functions

$f(t) = \langle g(t), h(t), \dots \rangle$: function with vector values.



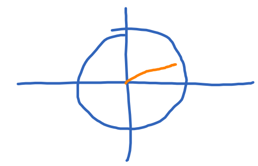
$\lim_{t\to t_0} f(t) = A$ iff $\begin{cases} g(t) \to A_x \\ h(t) \to A_y \\ \dots \end{cases}$

It is reasonable to define that :

$$\lim_{t \to t_0} f(t) = \left\langle \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t), \dots \right\rangle.$$

Ex: Find $\lim_{t \to 0} r(t)$

$$r(t) = \frac{1}{t} \langle \sin t, t \rangle$$



$$r(t) = \frac{1}{t} \langle \cos t, \sin t \rangle$$

Derivative

If $f(t) = \langle g(t), h(t), \dots \rangle$ then

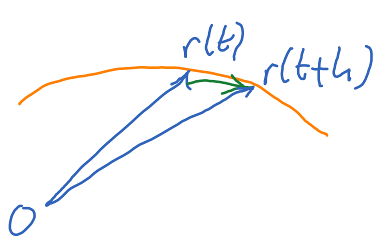
$$f'(t) = \langle g'(t), h'(t), \dots \rangle.$$

Why? $$f'(t) = \lim_{a \to 0} \frac{f(t+a) - f(t)}{a} = \lim_{a \to 0} \left\langle \frac{g(t+a) - g(t)}{a}, \dots \right\rangle$$

$$= \langle g'(t), h'(t), \dots \rangle$$

Ex: $r(t) = \langle t\cos t, t\sin t, t \rangle$

$r'(t) = \dots$



$r'(t)$ is a direction vector of the tangent vector.

Unit tangent vector:

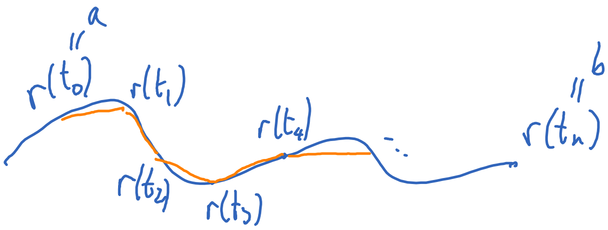
$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Ex:

$$r(t) = \langle e^t, \sin t, t \rangle$$

Find the tangent line at point $(1, 0, 0)$.

Length



$$\text{length} \approx \underbrace{|r(t_1) - r(t_0)|}_{\sim |r'(t_0)\Delta t|} + \underbrace{|r(t_2) - r(t_1)|}_{\sim |r'(t_1)\Delta t|} + \underbrace{|r(t_3) - r(t_2)|}_{\sim |r'(t_2)\Delta t|} + \cdots$$

$$\approx \sum_k |r'(t_k)| \Delta t$$

$$\boxed{\text{length} = \int_a^b |r'(t)|\, dt}$$

Ex:

$$r(t) = \langle t, 3\cos t, 3\sin t \rangle, \quad 0 \le t \le 4$$

What is the length of the curve?

$$\int_0^4 \sqrt{t^2 + 9}\, dt$$

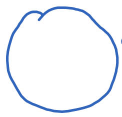
$$t = 3\tan\theta$$

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

Curvature

$$\kappa = \left| \frac{dT}{ds} \right|$$

"rate of change" of the unit tangent vector

 ← the circle has a constant curvature

$$\kappa = \frac{|T'(t)|}{|r'(t)|}$$

$$r(t) = \langle R\cos t, R\sin t, 0 \rangle$$

$$r'(t) = \langle -R\sin t, R\cos t, 0 \rangle$$

$$T(t) = \frac{r'(t)}{R} = \langle -\sin t, \cos t, 0 \rangle$$

$$T'(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$|T'(t)| = 1$$

$$|r'(t)| = R$$

$$\kappa = \frac{1}{R}$$

If R is large, κ is small.

If R is small, κ is large.

Another formula (equivalent):

$$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

(see textbook for the proof)