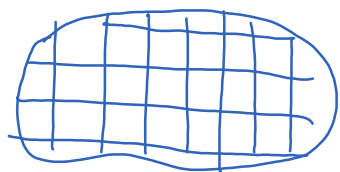


Lecture 36

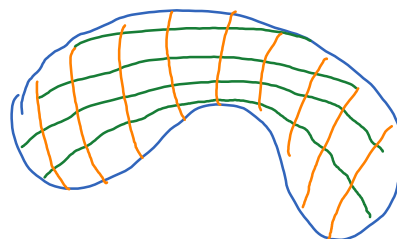
Wednesday, April 7, 2021 2:34 PM

- * Prayer
- * Spiritual thought
- * Answering questions ----

Surfaces



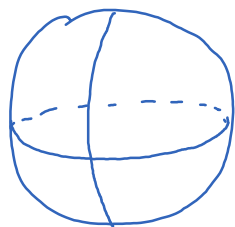
$(u, v) \in D$



$r(u, v) \in S$

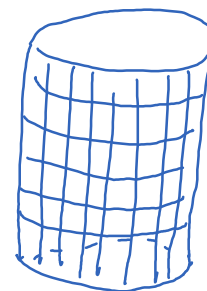
Ex: 1) The sphere $x^2 + y^2 + z^2 = 1$

$$r: \begin{cases} x = \sin \phi \cos \theta \\ y = \sin \phi \sin \theta \\ z = \cos \phi \end{cases} \quad \begin{array}{l} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{array}$$



2) The cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$

$$r: \begin{cases} x = \cos \theta \\ y = \sin \theta \\ z = z \end{cases} \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 1 \end{array}$$



3) The Mobius strip:

$$\begin{cases} x = 2 \cos \theta + r \cos \frac{\theta}{2} \\ y = 2 \sin \theta + r \cos \frac{\theta}{2} \\ z = r \sin \frac{\theta}{2} \end{cases} \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ \frac{1}{2} \leq r \leq \frac{1}{2} \end{array}$$

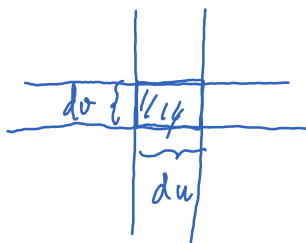
4) The torus:

$$\begin{cases} x = (a + r \cos \phi) \cos \theta \\ y = (a + r \cos \phi) \sin \theta \\ z = r \sin \phi \end{cases} \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq 2\pi \end{array}$$

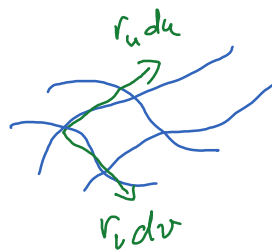
(one can take $a=2, r=1$)



Area of a surface



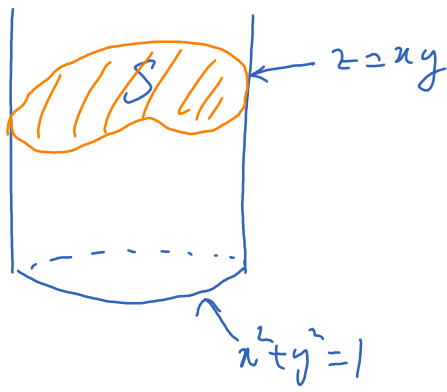
old area = $du dv$



$$\begin{aligned} \text{new area} &= |r_u du \times r_v dv| \\ &= |r_u \times r_v| du dv \end{aligned}$$

$$\text{Area of surface } S = \iint_D |r_u \times r_v| dA$$

E₂



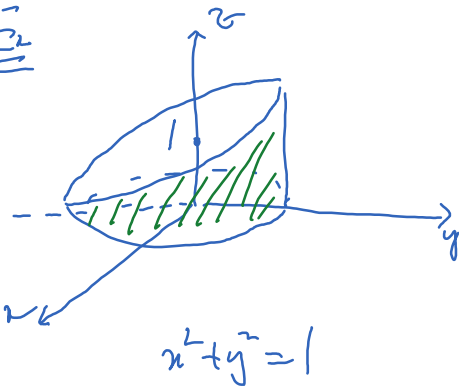
$$r: \begin{cases} x = u \\ y = v \\ z = uv \end{cases}$$

$(x, y) \in D$
 ↑
 unit disc

$$r_x = \langle 1, 0, y \rangle, \quad r_y = \langle 0, 1, x \rangle$$

$$\text{Area} = \iint_D \sqrt{x^2 + y^2 + 1} \, dA = \int_0^1 \int_0^{2\pi} \sqrt{r^2 + 1} \, r \, d\theta \, dr = \dots$$

E₂



$$S: \begin{cases} x = \cos \theta \\ y = \sin \theta \\ z = z \end{cases} \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq \sin \theta + 1 \end{array}$$

D

cut by the planes $z = |y+1|$ and $z = 0$

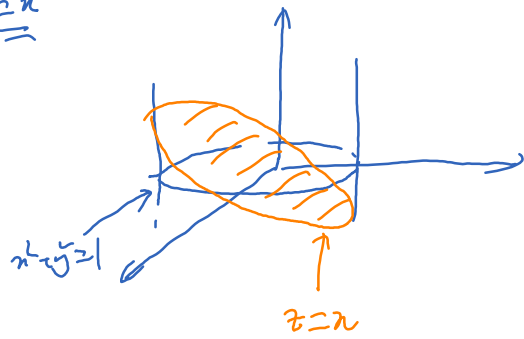
$$\text{Area} = \iint_D |r_\theta \times r_z| \, dA = \iint_D \underbrace{|\langle -\sin \theta, \cos \theta, 0 \rangle \times \langle 0, 0, 1 \rangle|}_{|\langle \cos \theta, \sin \theta, 0 \rangle|} \, dA = \text{area}(D)$$

$$\begin{array}{ccc} -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{array}$$

Another method:

$$\text{area} = \int_{\text{circle}} \text{height} \, ds = \int_C (y+1) \, ds = \int_0^{2\pi} (\sin \theta + 1) \underbrace{|r'(\theta)|}_{1} \, d\theta$$

Ex



Two ways;

$$R_1: \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = r \cos \theta \end{cases} \quad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$R_2: \begin{cases} x = x \\ y = y \\ z = x \end{cases} \quad \begin{array}{l} (x, y) \in D \\ \uparrow \\ \text{disc} \end{array}$$

$$\text{Area} = \int_0^{2\pi} \int_0^1 \left| \frac{\partial R_1}{\partial r} \times \frac{\partial R_1}{\partial \theta} \right| dr d\theta$$

$$= \iint_D \left| \frac{\partial R_2}{\partial x} \times \frac{\partial R_2}{\partial y} \right| dA = \int_0^{2\pi} \int_0^1 \left| \frac{\partial R_2}{\partial x} \times \frac{\partial R_2}{\partial y} \right| r dr d\theta$$

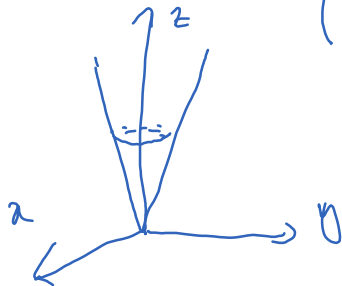
* Integral over a surface

Scalen: $\iint_S f(x, y, z) dS = \iint_D f(r(u, \theta)) \underbrace{|r_u \times r_\theta|}_{\text{area of a patch}} dA$

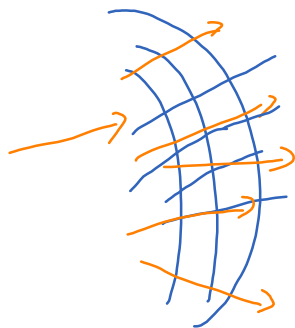
Ex

$$\iint_S xyz dS$$

$$S: \begin{cases} x = u \cos v \\ y = u \sin v \\ z = u \end{cases} \quad \begin{array}{l} 0 \leq u \leq 1 \\ 0 \leq v \leq \frac{\pi}{2} \end{array}$$



Flux



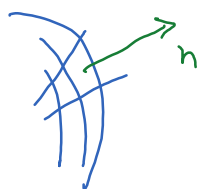
F : vector field

S : surface

the flux of F across S is

$$\iint_S F \cdot n \, dS = \iint_S \vec{F} \cdot d\vec{S}$$

↑
unit normal
vector on the
surface



$$\iint_S F \cdot n \, dS = \iint_D F \cdot \frac{r_u \times r_v}{|r_u \times r_v|} |r_u \times r_v| \, dA$$

$$= \iint_D F \cdot (r_u \times r_v) \, dA$$

Ex:

$$F = \langle x, y, z \rangle$$

$$S: \text{plane } x+y+z=1$$

