Lecture 7

Wednesday, January 27, 2021 2:05 PM

Ex: Find $\lim_{t \to 0} r(t)$

$$r(t) = \frac{1}{t} \langle \sin t, t \rangle$$

$$r(t) = \frac{1}{t} \langle \cos t, \sin t \rangle$$

Derivative

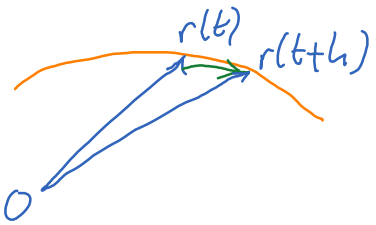
If $f(t) = \langle g(t), h(t), \ldots \rangle$ then

$$f'(t) = \langle g'(t), h'(t), \ldots \rangle.$$

Geometric meaning

$$r(t) = \langle t\cos t, t\sin t, t \rangle$$

$$r'(t) = \ldots.$$



$r'(t)$ is a direction vector of the tangent vector.

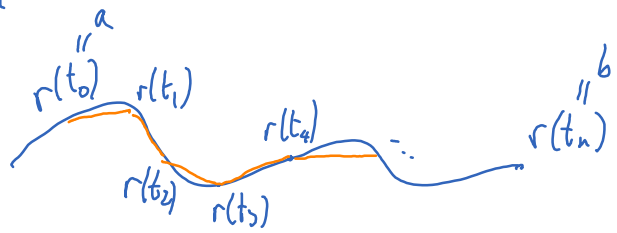
Unit tangent vector:

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Ex: $r(t) = \langle e^t, \sin t, t \rangle$

Find the tangent line at point $(1, 0, 0)$.

Length



$$\text{length} \approx \underbrace{|r(t_1)-r(t_0)|}_{\sim |r'(t_0)\Delta t|} + \underbrace{|r(t_2)-r(t_1)|}_{\sim |r'(t_1)\Delta t|} + \underbrace{|r(t_3)-r(t_2)|}_{\sim |r'(t_2)\Delta t|} + \cdots$$

$$\approx \sum_k |r'(t_k)|\,\Delta t$$

$$\boxed{\text{length} = \int_a^b |r'(t)|\,dt}$$

Ex:

$$r(t) = \langle t, 3\cos t, 3\sin t\rangle \quad , \quad 0 \le t \le 4$$

What is the length of the curve?

$$\int_0^4 \sqrt{t^2+9}\,dt$$

$t = 3\tan\theta$

$\tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$

Curvature $\quad \kappa = \left| \frac{dT}{ds} \right|$

"rate of change" of the unit tangent vector

← the circle has a constant curvature

$r(t) = \langle R\cos t, R\sin t, 0 \rangle$

$r'(t) = \langle -R\sin t, R\cos t, 0 \rangle$

$\kappa = \dfrac{|T'(t)|}{|r'(t)|}$

$T(t) = \dfrac{r'(t)}{R} = \langle -\sin t, \cos t, 0 \rangle$

$T'(t) = \langle -\cos t, -\sin t, 0 \rangle$

$|T'(t)| = 1$

$|r'(t)| = R$

$\kappa = \dfrac{1}{R}$

If R is large, κ is small.

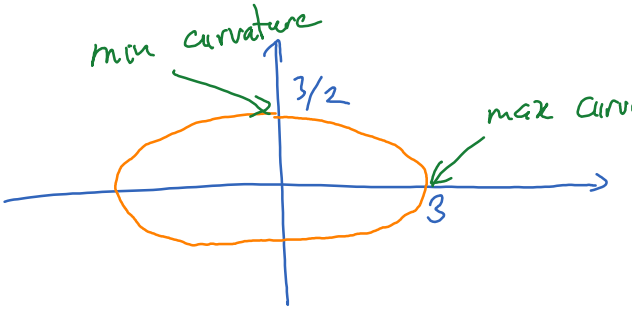
If R is small, κ is large.

Another formula (equivalent):

$$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

(see textbook for the proof)

Ex Ellipse $x^2+4y^2=9$. Find the curvature at $(3,0,0)$ and $(0,\frac{3}{2},0)$.



$$\begin{cases} x = 3\cos t \\ y = \frac{3}{2}\sin t \\ z = 0 \end{cases}$$

Then $r'(t) = \langle -3\sin t, \frac{3}{2}\cos t, 0\rangle$

$r''(t) = \langle -3\cos t, -\frac{3}{2}\sin t, 0\rangle$

$r'(t)\times r''(t) = \langle 0, 0, \frac{9}{2}\rangle \rightsquigarrow |r'(t)\times r''(t)| = \frac{9}{2}$

$|r'(t)| = \sqrt{9\sin^2 t + \frac{9}{2}\cos^2 t} = \frac{3}{\sqrt{2}}\sqrt{\frac{9}{2}\sin^2 t + \frac{9}{2}}$

$$\text{Curvature} = \frac{9/2}{\left(\frac{9}{2}\sin^2 t + \frac{9}{2}\right)^{3/2}}$$

$$\frac{9/2}{\left(\frac{9}{2}+\frac{9}{2}\right)^2} \le K \le \frac{9}{2}\Big/\left(\frac{9}{2}\right)^{3/2}$$