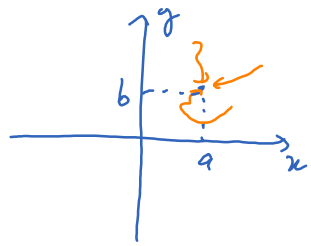
Lecture 13

Friday, February 4, 2022 12:29 AM

* Prayer
* Spiritual thought

Limit

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



How do we check if the limit of a given function exists?

Continuous functions are functions such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$: simply plug in (a,b) for (x,y)

This is quite a wide class of functions:

- constant functions
- polynomials : $x, y, x^2y, yz+zx^2, xyz, \dots$
- exponential : $e^x, e^y, e^z, e^{xy}, e^{x^2+yz}, \dots$
- trigonometric : $\sin(e^x), \cos(x+y), \dots$
- sum / product / composition of two cont. is a cont. function
- quotient of two cont. function is a continuous function (as long as the denominator is nonzero)

Ex $\lim_{(x,y)\to(1,-1)} \frac{\sin(x+y)}{x-y} = \frac{\sin(1-1)}{1-(-1)} = 0$

* Tricky situation : indeterminate $\frac{0}{0}, \frac{\infty}{\infty}, 1^{\infty}, 0^{0}$.

Strategy : guess the limit. Then either show that the limit exists or does not exist.

Ex
$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = ?$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim \frac{x^3}{x^2+y^2} - \lim \frac{y^3}{x^2+y^2}$$

Sandwich rule :

$$\left| \frac{x^3}{x^2+y^2} - 0 \right| = \frac{|x^3|}{x^2+y^2} = \frac{x^2}{x^2+y^2}|x| \leq |x| \longrightarrow 0$$

$$\left| \frac{y^3}{x^2+y^2} - 0 \right| = \frac{|y^3|}{x^2+y^2} = \frac{y^2}{x^2+y^2}|y| \leq |y| \longrightarrow 0$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0-0 = 0$$

Ex $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2-y^2} = ?$

Both numerator and denominator go to zero as $(x,y)\to(0,0)$. The speed of the numerator is determined only by x. The speed of the denominator is determined by both x and y. One may be able to choose y such that x^2-y^2 is very small compared to x^3.

Choose $y = x + x^3$. Then $x^2 - y^2 = x^2 - (x+x^3)^2 = -2x^4 - x^6$.

$$\frac{x^3}{x^2-y^2} = \frac{1}{-2x-x^3} \longrightarrow \text{DNE} \quad \text{as } x \to 0.$$

Conclusion: limit doesn't exist.