Lecture 17

Saturday, February 12, 2022 11:49 PM

* Prayer
* Spiritual thought
* Differential:

$f(x)$ $\qquad x \to x + \Delta x$

$$f(x) \to f(x+\Delta x) \approx f(x) + \underbrace{f'(x)\Delta x}_{\Delta f}$$

$$df = f'\,dx$$

$f(x,y)$ $\qquad x \to x+\Delta x$

$y \to y + \Delta y$

$$f(x,y) \to f(x+\Delta x, y+\Delta y) = f(x,y) + \overbrace{f(x+\Delta x, y) - f(x,y)}^{\approx f_x(x,y)\Delta x}$$

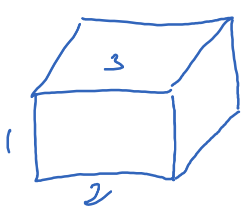
$$+ \underbrace{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)}_{\approx \Delta y\, f_y(x+\Delta x, y)}$$

$$\approx \Delta y\, f_y(x,y)$$

$$\Delta f \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy$$

Ex



Make a rectangular box with side lengths 1, 2, 3 ft

Allow error: $\frac{1}{4}$ in (on each side)

what is the error the volume?

$$f(x,y,z) = xyz$$

$$f(1,2,3) = 1(2)(3) = 6$$

$$f(1+\Delta x, 2+\Delta y, 3+\Delta z) \approx f(1,2,3) + f_x(1,2,3)\Delta x$$
$$+ f_y(1,2,3)\Delta y$$
$$+ f_z(1,2,3)\Delta z$$

$$\Delta f \approx 2(3)\Delta x + 1(3)\Delta y + 1(2)\Delta z$$
$$= 6\Delta x + 3\Delta y + 2\Delta z$$
$$\leq 6 \times \frac{1}{48} + 3 \times \frac{1}{48} + 2 \times \frac{1}{48} = \frac{11}{48} \quad ft^3$$
$$= \frac{11}{48} \times 12^3 \quad in^3$$
$$\frac{11 \times 144}{4} = 11 \times 36 = 396 \ in^3$$

Chain rule

Calc I $\begin{cases} y = (4x^2)^3 \\ y' = 3(4x^2)'(4x^2)^2 = 3(8x)(4x^2)^2 = \dots \end{cases}$

$$z = x^2 y^3, \quad x = u+v, \quad y = u^2 - v^2$$

$u \to u + \Delta u \rightsquigarrow x \to x + \underbrace{x_u \Delta u}_{\Delta x}, \quad y \to y + y_u \Delta u$

$v \to v$

$z \to z + \underbrace{\Delta z}_{?}$

$$\Delta z \approx z_x \Delta x + z_y \Delta y = z_x x_u \Delta u + z_y y_u \Delta u = (z_x x_u + z_y y_u)\Delta u$$

$\rightsquigarrow z_u = z_x x_u + z_y y_u$

$$\rightsquigarrow \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}.$$

Ex:

$$z = x^2 y^3 \,, \quad x = u+v \,, \quad y = u^2 - v^2$$

$$z_u = z_x x_u + z_y y_u = 2xy^3(1) + 3x^2y^2 2u$$

$$= 2(u+v)(u^2-v^2)^3 + 3(u+v)^2(u^2-v^2)^2 2u$$

Ex: $f(x,y) = x^2 + xy + y^2$

Find $f_x(x^2,y)$

$\frac{\partial}{\partial x}\left[f(x^2,y)\right]$ (two ways)

$f_x = 2x+y \rightsquigarrow f_x(x^2,y) = 2x^2+y$

$f(x^2,y) = x^4 + x^2y + y^2 \rightsquigarrow \frac{\partial}{\partial x}\left(f(x^2,y)\right) = 4x^3+2xy$

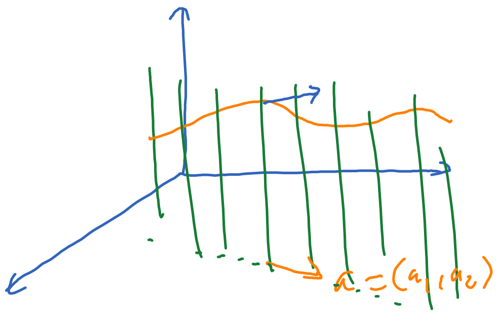
Another way:

$$\frac{\partial}{\partial x}\left[f(x^2,y)\right] = \frac{\partial f}{\partial x}(x^2,y)\underbrace{\frac{\partial (x^2)}{\partial x}}_{2x} + \frac{\partial f}{\partial y}(x^2,y)\underbrace{\frac{\partial y}{\partial x}}_{0}$$

$$= f_x(x^2,y)\,2x$$

Directional derivatives

$$D_a f(x,y) = \lim_{h \to 0} \frac{f(x+a_1h,\, y+a_2h) - f(x,y)}{h}$$



Other notations:

$$\nabla_a f(x,y)$$

$$\frac{\partial f}{\partial a}(x,y)$$

$$a_1 \underbrace{\frac{f(x+a_1h,\, y+a_2h) - f(x,\, y+a_2h)}{a_1 h}} + a_2 \underbrace{\frac{f(x,\, y+a_2h) - f(x,y)}{a_2 h}}$$

$\Downarrow h \approx 0$

$f_x(x,\, y+a_2h)$

$\Downarrow h \approx 0$

$f_x(x,y)$

$\Downarrow h \approx 0$

$f_y(x,y)$

Therefore, $D_a f(x,y) = a_1 f_x(x,y) + a_2 f_y(x,y)$

$$= \underbrace{\langle a_1, a_2 \rangle}_{a} \cdot \underbrace{\langle f_x, f_y \rangle}_{\nabla f} \quad \text{"gradient of } f\text{"}$$