Lecture 18

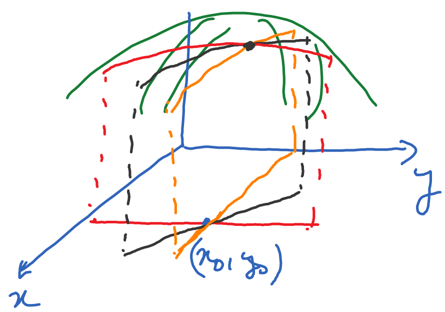
Tuesday, February 15, 2022 11:36 PM

* Prayer
* Spiritual thought
* Exercises on the chain rule

15 min

Directional derivatives

$f = f(x, y)$



$\frac{\partial f}{\partial x}(x_0, y_0) =$ slope of the orange curve at (x_0, y_0)

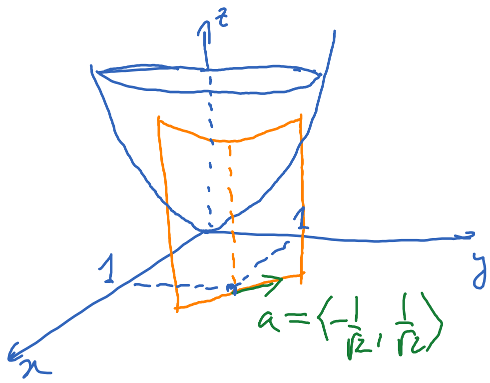
$\frac{\partial f}{\partial y}(x_0, y_0) =$ slope of the red curve at (x_0, y_0).

If we slice the surface by another plane through (x_0, y_0) and perpendicular to the plane xy, then we will have a different curve on the surface. The slope of this curve is called directional derivative

$\frac{\partial f}{\partial a}$, $D_a f$, $\nabla_a f$: equivalent notation

$$D_a f(x_0, y_0) = \lim_{h \to 0} \frac{f(x + ha_1, y + ha_2) - f(x, y)}{h}$$

Ex $f(x,y) = x^2 + y^2$



$$D_a f(1,1) = a \cdot \langle f_x(1,1), f_y(1,1) \rangle$$

$$\left.\begin{array}{l} f_x = 2x \\ f_y = 2y \end{array}\right\} \rightsquigarrow \left\{\begin{array}{l} f_x(1,1) = 2 \\ f_y(1,1) = 2 \end{array}\right.$$

$$D_a f(1,1) = -\frac{1}{\sqrt{2}}(2) + \frac{1}{\sqrt{2}}(2) = 0.$$

Vector $\langle f_x, f_y \rangle$ is called a gradient vector.

If we know the gradient vector at one point, we know the directional derivative wrt <u>any</u> direction at that point.

Directional derivative = rate of change of f along direction a.

* <u>Optimization problem</u>:

Which direction is the rate of change of f maximum?

$$D_a f(x_0,y_0) = \underbrace{a}_{\text{movable}} \cdot \underbrace{\nabla f(x_0,y_0)}_{\text{fixed}} \rightsquigarrow \text{maximum if } a = \frac{\nabla f}{|\nabla f|}$$

Ex

$$f(x,y) = 3 - x^2 - 2y^2$$

What is the direction at which f has maximum rate of change at $(1,1)$?

$$\nabla f(x,y) = \langle -2x, -4y \rangle$$

$$\nabla f(1,1) = \langle -2, -4 \rangle$$

$$a = \frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \frac{\langle -2, -4 \rangle}{\sqrt{(-2)^2 + (-4)^2}} = \frac{\langle -2, -4 \rangle}{\sqrt{20}} = \left\langle -\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle.$$

Chain rule:

$$g(u,v) = f(u^2+v^2, u-v)$$

g_u? g_v?

$$g_u = f_x x_u + f_y y_u = f_x 2u + f_y$$

$$g_v = f_x x_v + f_y y_v = f_x 2v - f_y$$

If $f_x(2,0) = 1$ and $f_y(2,0) = 2$ then

$$g_u = f_x(x,y)\, 2u + f_y(x,y)$$

$$g_u(1,1) = f_x(2,0)\, 2 + f_y(2,0) = 1(2) + 2 = 4.$$