Lecture 19

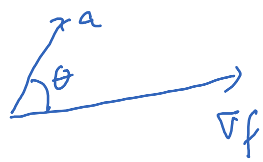
Thursday, February 17, 2022 11:20 PM

* Prayer

* Spiritual thought

* Directional derivative:

$$D_a f = a \cdot \nabla f$$



Fastest change in ∇f direction

Slowest change in the direction perpendicular to ∇f.

<u>Ex</u> $f(x,y) = x^2 + xy + y^2$

What direction does f change the most at $(1,2)$?

$$\nabla f = \langle 2x+y, x+2y \rangle$$

$$\nabla f(1,2) = \langle 4, 5 \rangle$$

$\rightsquigarrow$ fastest change in $\dfrac{\langle 4,5 \rangle}{|\langle 4,5 \rangle|} = \left\langle \dfrac{4}{\sqrt{41}}, \dfrac{5}{\sqrt{41}} \right\rangle$.

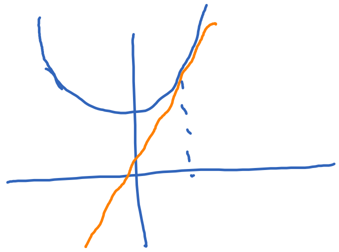
<u>rule</u>: gradient is always perpendicular to level sets.

p1 = ContourPlot[x^2+xy+y^2, {x,-1,1}, {y,-1,1}]

p2 = VectorPlot[{2x+y, 2y+x}, {x,-1,1}, {y,-1,1}]

p3 = Show[p1, p2]

Ex tangent plane at $(1,3)$ on the parabola $y = x^2+2$.



$f(x,y) = x^2+2-y$

Parabola = 0-level set of f

$\rightsquigarrow$ perpendicular to $\nabla f(1,3)$

$\nabla f = \langle 2x, -1\rangle \rightsquigarrow \nabla f(1,3) = \langle 2,-1\rangle$.

$\rightsquigarrow$ tangent vector $= \langle 1,2\rangle$

$\rightsquigarrow$ tangent line : $\begin{cases} x = 1+t \\ y = 3+2t \end{cases}$

Ex tangent plane to $x^2+y^2+2z^2=1$ at $\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$,

$f(x,y,z) = x^2+y^2+2z^2$

ellipsoid = 1-level set of f

$\rightsquigarrow$ perpendicular to $\nabla f\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$.

$\nabla f = \langle 2x, 2y, 4z\rangle \rightsquigarrow \nabla f\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) = \langle 1,1,-2\rangle$

$\rightsquigarrow$ eq. of tangent plane :

$$1\left(x-\tfrac{1}{2}\right)+1\left(y-\tfrac{1}{2}\right)+(-2)\left(z+\tfrac{1}{2}\right)=0$$

Optimization problem

f attains a local extremum at $(x_0, y_0, \dots)$ only if $\nabla f(x_0, y_0, \dots) = 0$.

If $\nabla f(x_0, y_0, \dots) = 0$ then $(x_0, y_0, \dots)$ is called a critical point of f.

Ex

$$f(x,y) = \frac{1}{4}x^4 - 2xy + x + y^2$$

$$\nabla f = \langle x^3 - 2y + 1,\ -2x + 2y \rangle$$

$$\nabla f = 0 \iff \begin{cases} x^3 - 2y + 1 = 0 \\ -2x + 2y = 0 \end{cases} \rightsquigarrow \begin{cases} x^3 - 2x + 1 = 0 \\ y = x \end{cases}$$

$$(x-1)(x^2 + x - 1) = 0$$

$$\rightarrow x = 1, \quad x = \frac{-1 \pm \sqrt{5}}{2}$$

We get 3 critical points:

$$(x,y) \in \left\{ (1,1), \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right), \left(\frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right) \right\}$$