Lecture 20

Tuesday, February 22, 2022 12:22 AM

Calc I derivative $\to$ min/max

$f(x) \to$ min/max

$f'(x) = 0 \to$ critical points

$x \in [a, b]$

Mult Calc: partial der $\to$ min/max

$f(x,y) \to$ min/max

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightsquigarrow \text{critical point of } f$$

Def A critical point of f is (x,y) such that $\nabla f(x,y) = \vec{0}$.

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2$

Find critical points.

$$\begin{cases} f_x = 3x^2 + y^2 - 6 = 0 & (1) \\ f_y = 2xy + 2y = 0 & (2) \end{cases}$$

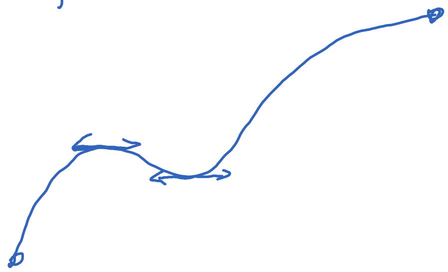
(2) $\rightsquigarrow 2y(x+1) = 0$

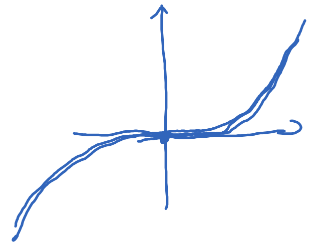
- $y = 0 \rightsquigarrow 3x^2 - 6 = 0 \rightsquigarrow x^2 = 2 \rightsquigarrow x = \pm\sqrt{2}$
- $x = -1 \rightsquigarrow 3(-1)^2 + y^2 - 6 = 0 \rightsquigarrow y = \pm\sqrt{3}$

(x,y):
- $(\sqrt{2}, 0)$
- $(-\sqrt{2}, 0)$
- $(-1, \sqrt{3})$
- $(-1, -\sqrt{3})$

} All critical points of f.

Calc I: $f'(x) = 0$





$f(x) = x^3$

$f' = 3x^2 \quad f'(0) = 0$

Mult Calc: $\nabla f(x,y) = 0$

How to decide if a critical point is local min/max?

Calc I: $f'(x) = 0$

$f''(x) > 0$ local min

$f''(x) < 0$ local max

$f''(x) = 0$ inconclusive

Mult Calc: $f(x,y)$

$f_{xx}, \ f_{xy} = f_{yx}, \ f_{yy}$

Second derivative test:

$f(x,y)$ has $\underbrace{\text{critical point at } (x_0, y_0)}_{\nabla f(x_0,y_0) = 0}$.

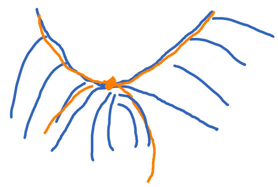
$D = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$

$= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

If $D > 0$ & $f_{xx}(x_0, y_0) > 0$ then (x_0, y_0) is a local min

If $D > 0$ & $f_{xx}(x_0, y_0) < 0$ then ...... local max.

If $D < 0$ then (x_0, y_0) is saddle point.



If $D = 0$: inconclusive.

If $D > 0$ & $f_{xx} = 0$: inconclusive.

Ex

$$f(x,y) = x^3 + xy^2 - 6x + y^2$$

$$(\sqrt{2}, 0),\ (-\sqrt{2}, 0),\ (-1, \sqrt{3}),\ (-1, -\sqrt{3})$$

$f_x = 3x^2 + y^2 - 6$ $\qquad$ $f_y = 2xy + 2y$

$f_{xx} = 6x$ $\qquad$ $f_{yy} = 2x + 2$

$f_{xy} = 2y$

$$D = f_{xx} f_{yy} - f_{xy}^2 = (6x)(2x+2) - (2y)^2$$

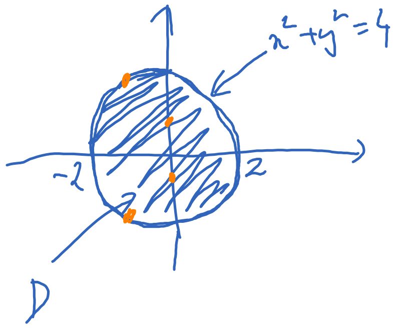
<u>Check $(\sqrt{2}, 0)$:</u>

$$D = (6\sqrt{2})(2\sqrt{2}+2) - \underbrace{(2(0))^2}_{0} > 0$$

$$f_{xx}(\sqrt{2}, 0) = 6\sqrt{2} > 0$$

$\Big\}$ $(\sqrt{2}, 0)$ is a local min.

global min/max ?



$$f(x,y) = x^3 + xy^2 - 6x + y^2$$

find min/max of f as $(x,y) \in D$.

Calc I: min/max $f(x)$, $x \in [a,b]$

$f(\sqrt{2},0) = ?$

$f(-\sqrt{2},0) = ?$

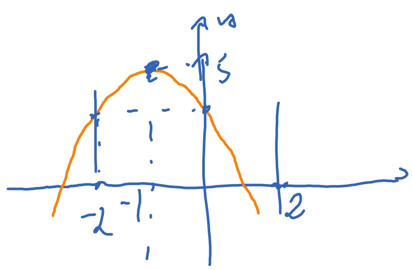
$f(-1,\sqrt{3}) = ?$

$f(-1,-\sqrt{3}) = ?$

On the boundary: $x^2+y^2=4$

$$y^2 = 4 - x^2$$

$$f(x,y) = x^3 + x(4-x^2) - 6x + 4 - x^2 = \underbrace{-x^2 - 2x + 4}_{\text{min/max}} \quad x \in [-2,2]$$



max $(x=-1) = 5$

min $(x=2) = -4$