Lecture 22

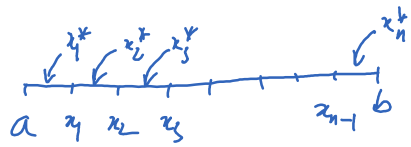
Thursday, February 24, 2022 10:33 PM

* Prayer

* Spiritual thought

Limit $\longrightarrow$ Derivative $\longrightarrow$ Applications of derivatives $\longrightarrow$ Integral

Calc I : $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (analytic definition)

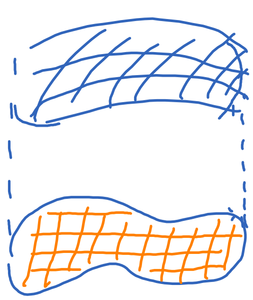


Geometrically, $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$.

Multivariable calc:

$$\underbrace{\iint_D f(x,y)\,dA}_{\text{double integral}}, \quad \underbrace{\iiint_E f(x,y,z)\,dV}_{\text{triple integral}}$$

Note: double/triple integrals don't have "orientation", unlike single integral.

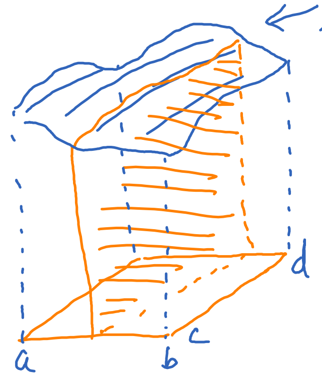


$$\iint_D f(x,y)\,dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i^*, y_j^*)\,\Delta A_{ij}$$

Geometrically, $\iint_D f(x,y)\,dA$ represents the volume under the surface $z = f(x,y)$

How to compute $\iint_D f(x,y)\,dA$ precisely?

The answer depends on the shape of D. The easiest case is when D is a rectangle $D = [a,b] \times [c,d]$.



$$\iint_D f(x,y)\,dA = \text{volume of the solid}$$

$$= \int_a^b S(x)\,dx$$

where $S(x)$ is the area of the cross section of the solid at x.

$$S(x) = \int_c^d f(x,y)\,dy.$$

Therefore,

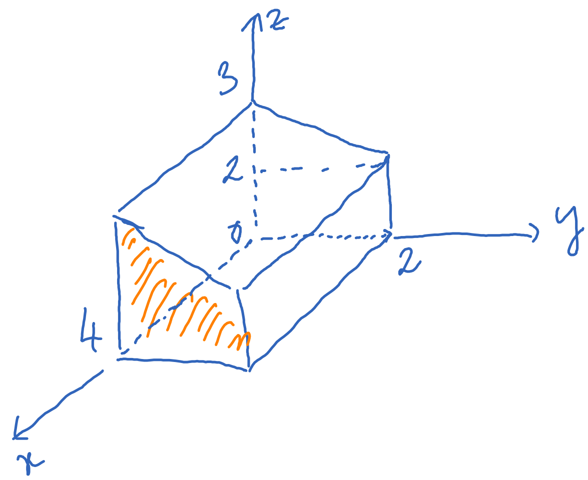
$$\underbrace{\iint_D f(x,y)\,dA}_{\text{double integral}} = \underbrace{\int_a^b \int_c^d f(x,y)\,dy\,dx}_{\text{iterated integral}} = \underbrace{\int_c^d \int_a^b f(x,y)\,dx\,dy}_{\text{iterated integral}}$$

Ex:

$$\iint_{[-1,1]\times[0,2]} (x^2 + xy)\,dA = \int_{-1}^{1}\int_0^2 (x^2+xy)\,dy\,dx = \int_{-1}^{1} \left(x^2 y + \frac{xy^2}{2}\right)\Bigg|_{y=0}^{y=2} dx$$

$$= \int_{-1}^{1} (2x^2 + 2x)\,dx = \left(\frac{2x^3}{3} + x^2\right)\Bigg|_{-1}^{1} = \frac{4}{3}.$$

Ex Fridge of this shape:



what is the volume of this fridge?

Two ways:

(1) Volume = area of the orange trapezoid × 4

$$= \frac{3+2}{2} \times 2 \times 4 = 20$$

(2) Let $f(x,y)$ be a function whose graph is given by the top lid of the fridge. Then

$$\text{volume} = \iint_{[0,4]\times[0,2]} f(x,y)\,dA = \int_0^4 \int_0^2 f(x,y)\,dy\,dx$$

$$f(x,y) = 3 - \frac{y}{2} \rightsquigarrow \text{volume} = \int_0^4 \int_0^2 \left(3 - \frac{y}{2}\right) dy\,dx$$

$$= \int_0^4 \left(3y - \frac{y^2}{4}\right)\Big|_{y=0}^{y=2} dx$$

$$= \int_0^4 5\,dx = 20$$

Example:
Solid given by -1-x^2-y^2 <= z <= x^2-y^2, -2 <= x,y <=2
What is the volume?

Mathematica:
$R = \text{ImplicitRegion}[-2 - x\text{^}2 - y\text{^}2 \leq z \leq x\text{^}2 - y\text{^}2 \&\& -2 \leq x \leq 2 \&\& -2 \leq y \leq 2, \{x, y, z\}]$
$\text{Region}[R, \text{AspectRatio} \to 1]$