Lecture 27

Thursday, March 10, 2022 8:47 PM

* Prayer
* Spiritual thought

Change of variables

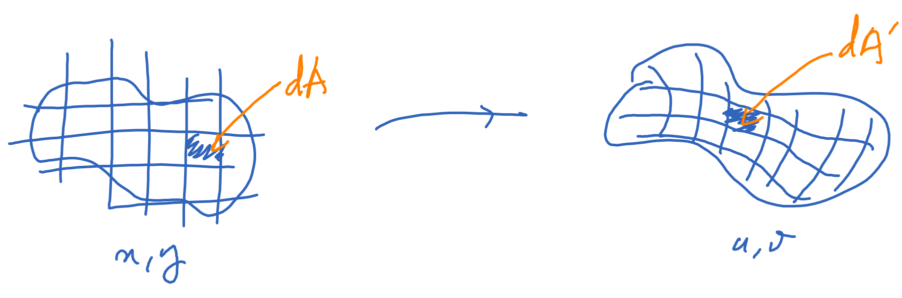
- Calc I: $\int_a^b f(x)\,dx \longrightarrow$ change variables when function $f(x)$ is complicated

integral has two components: integrand, region

- Multivariable calculus:

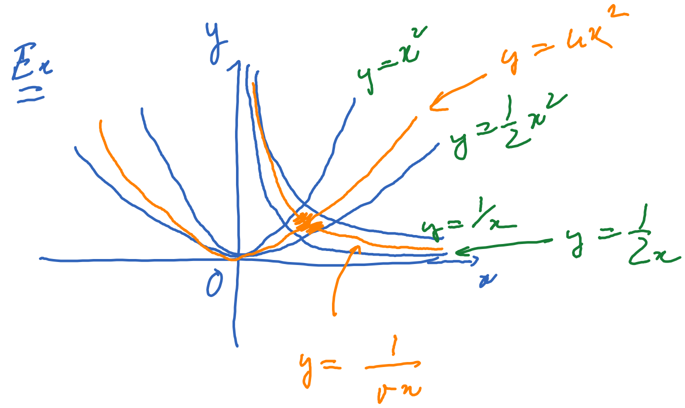
most of the times, we use change of variables when the region is bad (not rectangular)

$$\iint_D f(x,y)\,dA = \iint_{D'} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA'$$



$$\frac{dA}{dA'} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \qquad \left(\frac{\text{old}}{\text{new}}\right)$$

Ex



$\frac{1}{2} \le u \le 1, \quad 1 \le v \le 2.$

$$\begin{cases} y = ux^2 \\ y = \frac{1}{vx} \end{cases} \rightsquigarrow \begin{cases} x = \frac{1}{(uv)^{1/3}} = (uv)^{-1/3} \\ y = \frac{u^{1/3}}{v^{2/3}} = u^{1/3} v^{-2/3} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} -\frac{1}{3}u^{-4/3}v^{-1/3} & -\frac{1}{3}u^{-1/3}v^{-4/3} \\ \frac{1}{3}u^{-2/3}v^{-2/3} & -\frac{2}{3}u^{1/3}v^{-5/3} \end{pmatrix}$$

$$= \frac{2}{9}u^{-1}v^{-2} + \frac{1}{9}u^{-1}v^{-2} = \frac{1}{3}u^{-1}v^{-2}$$

$$\iint_D xy\,dA = \iint_{D'} (uv)^{-1/3} u^{1/3} v^{-2/3} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA'$$

$$= \iint_{D'} v^{-1} \frac{1}{3} u^{-1} v^{-2}\, dA'$$

$$= \int_{1/2}^{1}\int_{1}^{2} \frac{1}{3} u^{-1} v^{-3}\, dv\, du = \dots$$

Ex
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Triple integral

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV'$$

$$(x,y,z) \longrightarrow (u,v,w)$$

$$f(x,y,z) \longrightarrow g(u,v,w) \qquad \frac{dV}{dV'} = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

$$\underbrace{E}_{\text{old}} \longrightarrow \underbrace{E'}_{\text{new}}$$

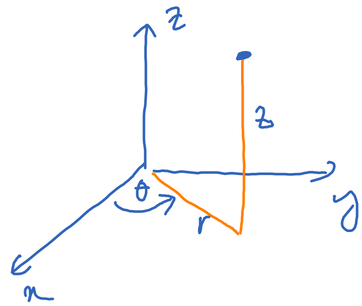
Two famous examples of change of variables in 3D are:

$\begin{cases} \text{cylindrical coordinates} \\ \text{spherical coordinates} \end{cases}$

* Cylindrical coordinates:

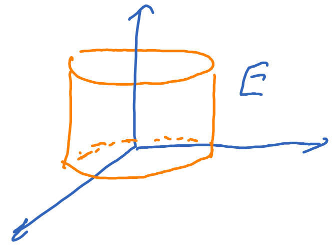
$$(x,y,z) \longrightarrow (r,\theta,z)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



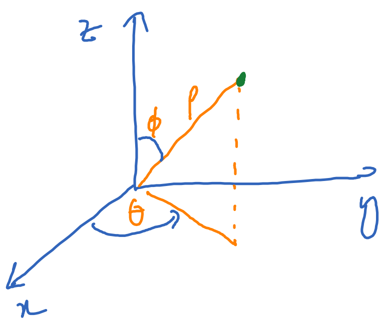
This change of variables is suitable for cylindrical solids.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

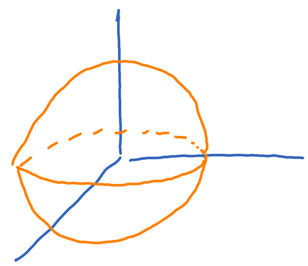


* Spherical coordinates:

$$(x,y,z) \longrightarrow (\rho,\theta,\phi)$$



This change of coordinates is suitable for spherical solids.



In fact, GPS uses spherical coordinates to record positions.

θ: Longitude

ϕ: 90° – Lattitude