Lecture 33

Saturday, March 26, 2022 9:57 PM

* Prayer

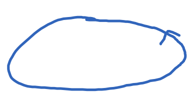
* Spiritual thought

* Line integral of vector field:

$$\left.\begin{array}{l} F = (P, Q) \\ dr = (dx, dy) \end{array}\right\} \quad F \cdot dr = P\,dx + Q\,dy = P x'\,dt + Q y'\,dt$$

Ex

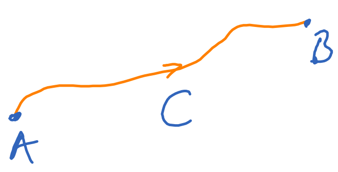
$$\int_C y\,dx - x\,dy = \int_C \underbrace{(y, -x)}_{F} \cdot \underbrace{(dx, dy)}_{dr} = \int_0^{2\pi} \left[3\sin t\,(-2\sin t) - 2\cos t\,(3\cos t)\right] dt$$



$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \longrightarrow \begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases} \quad 0 \le t \le 2\pi$$

* Fundamental theorem of calculus:

$$\int_C \nabla\phi \cdot dr = \phi(B) - \phi(A)$$



Note: $\displaystyle\int_C f_x\,ds \neq f(B) - f(A)$

The fundamental thm of calc. only works with the line integral of a vector field.

Note every vector field is the gradient of a scalar function.

$\underline{\underline{Ex}}$ $F(x,y) = (y, -x)$

If $F = \nabla\phi$ for some function ϕ then $\phi_x = y$ and $\phi_y = -x$.

Then $\phi_{xy} = 1$ and $\phi_{yx} = -1$. This is impossible!

<u>Def</u> A vector field F which is the gradient of some function ϕ is called a <u>conservative</u> vector field. The function ϕ is called the <u>potential</u> function of F.

$\underline{\underline{Ex}}$: $F(x,y) = (\underbrace{e^x \cos y}_{P}, \underbrace{-e^x \sin y + 1}_{Q})$

Is F a conservative vector field? If so, find a potential function.

<u>Step 1</u>: Check if $P_y = Q_x$.

$e^x \cos y + y$

$$\left. \begin{aligned} P_y &= -e^x \sin y \\ Q_x &= -e^x \sin y \end{aligned} \right\} P_y = Q_x$$

<u>Step 2</u>

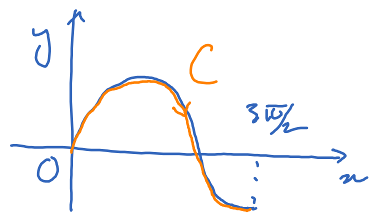
Find ϕ such that $P = \phi_x$ and $Q = \phi_y$

- $e^x \cos y = \phi_x \rightsquigarrow \phi = e^x \cos y + f(y)$
- $\phi_y = -e^x \sin y + f'(y) \rightsquigarrow f'(y) = 1 \rightsquigarrow f(y) = y + C$

<u>Step 3</u> $\phi = e^x \cos y + y + C$

Ex $F(x,y) = (y, x)$

$C: \; r(t) = (t, \sin t), \quad 0 \le t \le \frac{3\pi}{2}$

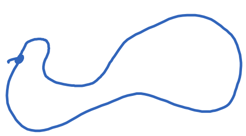


$$\int_C F \cdot dr = \int_C \nabla\phi \cdot dr \qquad \text{with } \phi = xy$$

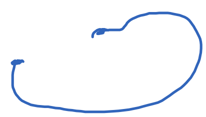
$$= \phi\left(\frac{3\pi}{2}, -1\right) - \phi(0,0) = -\frac{3\pi}{2}.$$

* Green's theorem

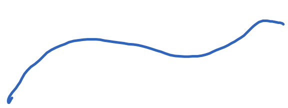
This theorem convert a line integral of a vector field over a closed curve into a double integral over the region enclosed by the curve.



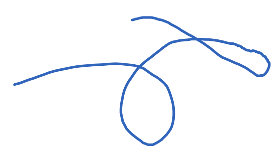
closed curve



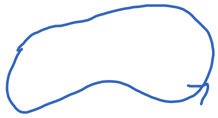
not a closed curve



simple curve

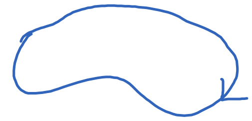


not a simple curve



close, simple curve

positively oriented

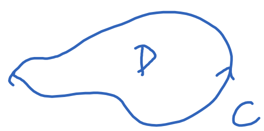


close, simple curve

negatively oriented

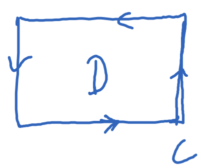
Green's theorem

$$\int_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



where C is a closed, simple curve, positively oriented.

Why is this true?



$$\int_C F \cdot dr = \iint_D (Q_x - P_y)\, dA$$