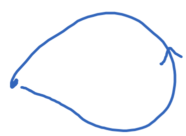
Lecture 34

Wednesday, March 30, 2022 3:02 PM

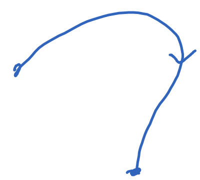
* Prayer
* Spiritual thought

$$\int_C F \cdot dr = \text{total circulation of the flow } F \text{ along the curve } C.$$

Interesting case: C is a closed curve



closed

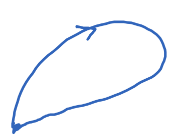


not closed

If F is conservative and C is a closed curve then $\int_C F \cdot dr = 0$.

why?

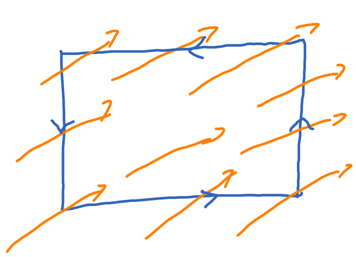
$$\int_C F \cdot dr = \int_C \nabla\phi \cdot dr = \phi(B) - \phi(A) = 0$$



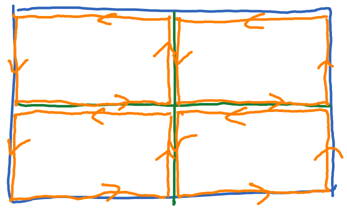
C

$A \equiv B$

Consider the case when C is the border of the rectangle:



$$\int_C F \cdot dr = \text{total circulation}$$

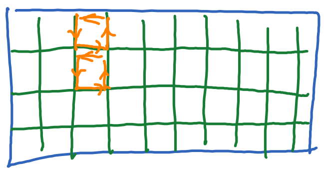


total circulation = circulation in rectangle 1

+ " " " 2

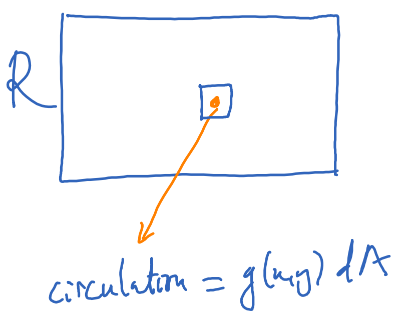
+ " " " 3

+ " " " 4.



total circulation = sum of the local circulations.

Let $g(x,y)$ = circulation density (over unit area)



Circulation in the cell dA around (x,y) is

$$g(x,y)\,dA$$

Total circulation = $\iint_R g(x,y)\,dA$

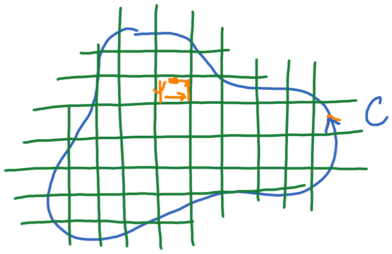
Therefore, $$\int_C F\cdot dr = \iint_R g(x,y)\,dA$$

The question is: how to find the circulation density from F?

Green's theorem: If $F = (P, Q)$ then $g(x,y) = Q_x - P_y$.

Note: if F is conservative then $g(x,y) = (\phi_y)_x - (\phi_x)_y = 0$.

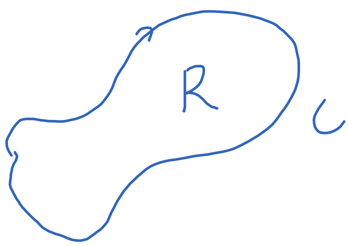
Green's theorem also applies for more general curves.



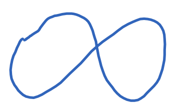
<u>Green's theorem</u>: Let C be a closed, simple, positively oriented curve.

Then

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$

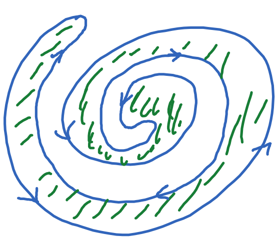


simple curve

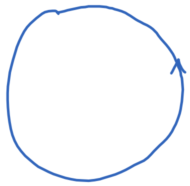


not simple curve

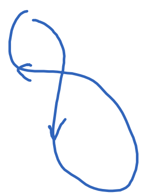
A simple curve doesn't intersect itself.



positively orientation is the orientation such that if you walk along the curve, the bounded region is always on your left



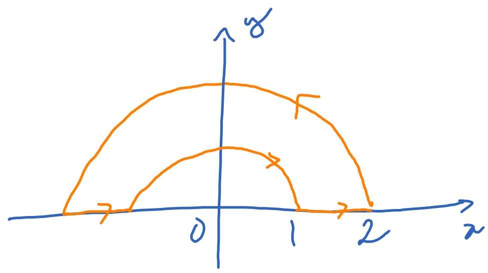
On a circle, positive orientation is the counterclockwise orientation.



It is not possible to give a single orientation on a non-simple curve.

<u>Ex</u>

$$\int_C y^2\,dx + 3xy\,dy = ?$$



If you want to find this line integral using parametrization, you have to break the curve into 4 pieces.

Using Green's thm: $P = y^2$, $Q = 3xy$

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA = \iint_R (3y - 2y)\,dA = \iint_R y\,dA$$

$$= \int_0^{\pi}\int_1^2 r\sin\theta \; r\,dr\,d\theta = \cdots$$