Lecture 35

Thursday, March 31, 2022 9:34 PM

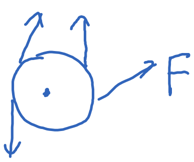
* Prayer
* Spiritual thought

<u>Green's thm:</u>

$$\underbrace{\int_C P\,dx + Q\,dy}_{\text{total circulation}} = \iint_R \underbrace{(Q_x - P_y)}_{\text{circulation density}}\, dA$$

Sometimes, people write $\ointctrclockwise$ or $\ointclockwise$ to indicate the orientation of the curve C.

Why is the circulation density equal to $Q_x - P_y$?



$$\int_C F \cdot dr = \int_0^{2\pi} -P(x,y)\,\varepsilon \sin\theta \, d\theta + \int_0^{2\pi} Q(x,y)\,\varepsilon\cos\theta\, d\theta$$

$$P(x,y) \approx P(0,0) + P_x(0,0)\,x + P_y(0,0)\,y$$

$$= P(0,0) + P_x(0,0)\,\varepsilon\cos\theta + P_y(0,0)\,\varepsilon\sin\theta$$

$$\int_0^{2\pi} -P(x,y)\,\varepsilon\sin\theta\, d\theta \approx \int_0^{2\pi} -\left[P(0,0) + P_x(0,0)\,\varepsilon\cos\theta + P_y(0,0)\,\varepsilon\sin\theta\right]\varepsilon\sin\theta\, d\theta$$

$$\approx -\pi\varepsilon^2 P_y(0,0)$$

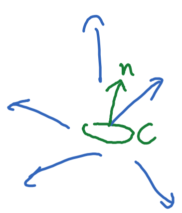
Similarly, $\displaystyle\int_0^{2\pi} Q(x,y)\,\varepsilon\cos\theta\, d\theta \approx \pi\varepsilon^2 Q_x(0,0)$

Thus, $\displaystyle\oint_C F\cdot dr \approx \pi\varepsilon^2\left(Q_x(0,0) - P_y(0,0)\right)$

$$\rightsquigarrow \frac{1}{\pi\varepsilon^2}\oint_C F\cdot dr \approx \underbrace{Q_x(0,0) - P_y(0,0)}_{\text{this must be the circulation density}}$$

The quantity $Q_x - P_y$ represents the "local" rotation of the vector field F at (x,y).

How about 3D?



$$\oint_C F\cdot dr \approx \varepsilon\pi^2\, n\cdot\omega$$

C: circle radius ε $\qquad$ n: "normal vector" of C

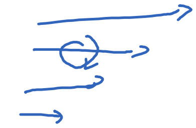
Thm: $\omega = \text{curl}\, F = \nabla\times F = \left(R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\right)$

$$\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ P & Q & R & P & Q \end{array}$$

curl F is the direction where F spins the most. Pinwheel spins most strongly when faces toward a certain direction.

curl F measures the local rotation.

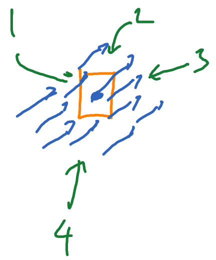
Ex $\quad F(x,y,z) = (y,0,0)$

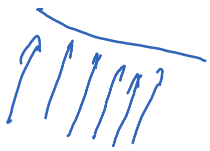


$\text{curl}\,F = (0-0,\ 0-0,\ -1\ ) = (0,0,-1)$

Divergence

$F = (P, Q)$





Suppose that the vector field is constant.

Flux going out through edge 1 is

$$-P(x,y)\,dy$$

Flux going out through edge 3 is

$$P(x+dx, y)\,dy$$

Flux going out through edge 2

$$Q(x, y+dy)\,dx$$

Flux going out through edge 4

$$-Q(x,y)\,dx$$

$$\text{Total flux out} = \underbrace{P(x+dx,y)\,dy - P(x,y)\,dy}_{= P_x(x,y)\,dx\,dy} + \underbrace{Q(x,y+dy)\,dx - Q(x,y)\,dx}_{= Q_y(x,y)\,dx\,dy}$$

$$= \underbrace{(P_x + Q_y)}_{\text{flux density, or "divergence"}}\,dx\,dy$$

$$\operatorname{div} F = \nabla \cdot F = P_x + Q_y$$

If $\operatorname{div} F < 0$ then the vector field tends to go toward (x,y). (sink)

If $\operatorname{div} F > 0$ then " " go away from (x,y). (source)