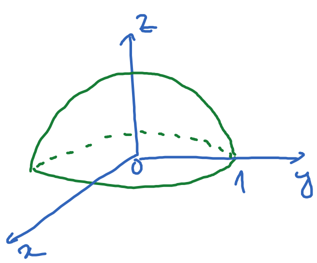
Lecture 38

Thursday, April 7, 2022 10:08 PM

* Prayer
* Spiritual thought

Recall: $$\iint_S f\,dS = \iint_R f(x(u,v), y(u,v), z(u,v))\,|r_u \times r_v|\,dA$$

Ex Integrate the function $f(x,y,z) = z$ over the upper half sphere $x^2+y^2+z^2=1$.



Parametrization:

$$\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \qquad \begin{array}{l} 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{array}$$

$$|r_\phi \times r_\theta| = \sin\phi$$

$$\iint_S f\,dS = \int_0^{2\pi}\int_0^{\pi/2} \cos\phi\,\sin\phi\,d\phi\,d\theta = \pi$$

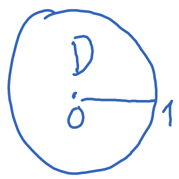
Another parametrization:

$$\begin{cases} x = x \\ y = y \\ z = \sqrt{1-x^2-y^2} \end{cases} \qquad r = (x, y, \sqrt{1-x^2-y^2})$$

$$\left.\begin{array}{l} r_x = \left(1, 0, \dfrac{-x}{\sqrt{1-x^2-y^2}}\right) \\ r_y = \left(0, 1, \dfrac{-y}{\sqrt{1-x^2-y^2}}\right) \end{array}\right\} \quad r_x \times r_y = \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right)$$

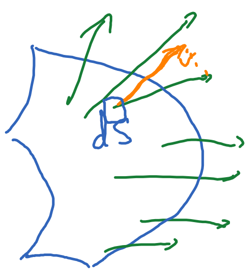
$$|r_x \times r_y| = \sqrt{\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1} = \frac{1}{\sqrt{1-x^2-y^2}}$$

Then $$\iint_S z\,dS = \iint_D \sqrt{1-x^2-y^2}\,\frac{1}{\sqrt{1-x^2-y^2}}\,dA = \iint_D dA = \text{area}(D) = \pi$$



Now we consider the second kind of surface integral:

$$\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{flux of } \vec{F} \text{ across the surface } S} \qquad \text{(integral of a vector field)}$$



$\vec{F}\cdot\vec{n}$ = the normal component of F

$\vec{F}\cdot\underbrace{\vec{n}\,dS}_{d\vec{S}}$ = flux of $\vec{F}$ across dS

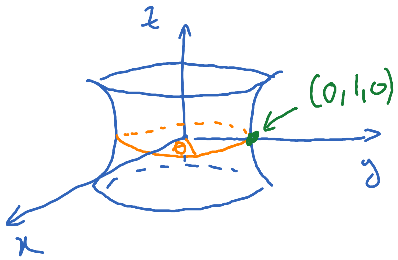
Thus, $$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

What is $\vec{n}$? (unit normal vector)

$$\vec{n} = \pm\frac{r_u\times r_v}{|r_u\times r_v|}$$ (plus or minus sign depending on the orientation of the surface)

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot\frac{[\pm(r_u\times r_v)]}{|r_u\times r_v|}\,|r_u\times r_v|\,dA = \iint_R \vec{F}\cdot[\pm(r_u\times r_v)]\,dA$$

Ex S is the surface obtained by rotating the curve $y = z^2+1$, $-1 \le z \le 1$ about the z-axis. Find the flux of the vector field



$F(x,y,z) = (0, 0, z)$ across the surface S in the "outward" direction.

A point on the surface is completely determined by a pair (z, θ):

$$S: \begin{cases} x = (z^2+1)\cos\theta \\ y = (z^2+1)\sin\theta \\ z = z \end{cases} \qquad \begin{matrix} -1 \le z \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$r(z,\theta) = ((z^2+1)\cos\theta, (z^2+1)\sin\theta, z)$$

$$r_z = (2z\cos\theta, 2z\sin\theta, 1)$$

$$r_\theta = (-(z^2+1)\sin\theta, (z^2+1)\cos\theta, 0)$$

$$r_z \times r_\theta = (-(z^2+1)\cos\theta, -(z^2+1)\sin\theta, 2z(z^2+1))$$

To see if this normal vector points "outward" or "inward", we check at a point on S.

At point $(0,1,0)$, we have $z=0$ and $\theta = \frac{\pi}{2}$.

$r_z \times r_\theta$ at this point is $\left(-(0^2+1)\cos\frac{\pi}{2}, -(0^2+1)\sin\frac{\pi}{2}, 2 0(0^2+1)\right)$

$= (0, -1, 0)$, which is a vector pointing inward.

Therefore, the normal vector pointing outward is

$$-r_z \times r_\theta = \left((z^2+1)\cos\theta,\ (z^2+1)\sin\theta,\ -2z(z^2+1) \right)$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_R (0,0,z)\cdot(-r_z \times r_\theta)\, dA \qquad R = [-1,1]\times[0,2\pi]$$

$$= \iint_R (0,0,z)\cdot\left((z^2+1)\cos\theta,\ (z^2+1)\sin\theta,\ -2z(z^2+1) \right) dA$$

$$= \iint_R -2z^2(z^2+1)\, dA = \int_0^{2\pi}\int_{-1}^{1} -2z^2(z^2+1)\, dz\, d\theta$$

$$= -4\pi \int_{-1}^{1} (z^4+z^2)\, dz = -4\pi \left(\frac{z^5}{5} + \frac{z^3}{3} \right)\Big|_{-1}^{1}$$

$$= -8\pi\left(\frac{1}{5}+\frac{1}{3}\right) = -\frac{64\pi}{15}$$