Lecture 7

Wednesday, January 19, 2022 10:26 AM

* Prayer

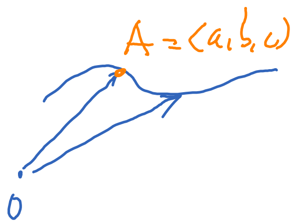
* Spiritual thought

Calculus I $\rightsquigarrow$ $\left\{\begin{array}{l}\text{limit}\\ \text{derivative}\\ \text{integral}\end{array}\right.$ of single-variabled functions

Calculus of Several Variables $\rightsquigarrow$ $\left\{\begin{array}{l}\text{limit}\\ \text{derivative}\\ \text{integral}\end{array}\right.$ of multivariable functions

Vector-valued function:

$r(t) = \langle x(t), y(t), z(t) \rangle$: curve in 3D.



Limit of this function as $t \to t_0$?

$$\lim_{t \to t_0} r(t) = A \iff |r(t) - A| \to 0 \text{ as } t \to t_0$$

Equivalently,

$$\lim_{t \to t_0} r(t) = A \iff \begin{cases} \lim_{t \to t_0} x(t) = a \\ \lim_{t \to t_0} y(t) = b \\ \lim_{t \to t_0} z(t) = c \end{cases}$$

Equivalently,

$$\lim_{t \to t_0} r(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle.$$

Ex: (a) $r(t) = \frac{1}{t}\langle t, \sin t\rangle$

(b) $r(t) = \frac{1}{t}\langle \cos t, \sin t\rangle$

Find $\lim\limits_{t\to 0} r(t)$.

$$\frac{1}{t}\langle t, \sin t\rangle = \left\langle \underset{\substack{\downarrow \\ 1}}{1}, \underset{\substack{\downarrow \\ 1}}{\frac{\sin t}{t}} \right\rangle \xrightarrow{t\to 0} \langle 1, 1\rangle$$

$$\frac{1}{t}\langle \cos t, \sin t\rangle = \left\langle \underset{\substack{\downarrow \\ \text{DNE}}}{\frac{\cos t}{t}}, \underset{\substack{\downarrow \\ 1}}{\frac{\sin t}{t}} \right\rangle \qquad \text{DNE}$$

Derivative

$$r(t) = \langle x(t), y(t), z(t)\rangle$$

What does $r'(t)$ mean? Consider two possible ways to define:

① $r'(t) = \langle x'(t), y'(t), z'(t)\rangle$

② $r'(t) = \lim\limits_{h\to 0} \dfrac{r(t+h) - r(t)}{h}$

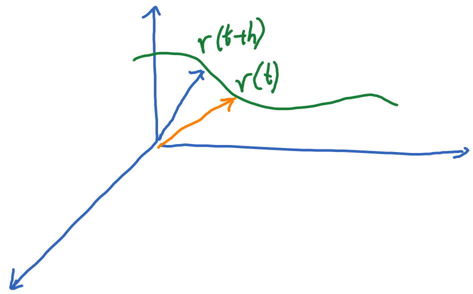
These two methods are in fact the same as each other.

$$\lim_{h\to 0} \frac{r(t+h) - r(t)}{h} = \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \dots, \dots \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \dots, \dots \right\rangle$$

$$= \langle x'(t), y'(t), z'(t)\rangle$$

How to interpret $r'(t)$ geometrically?



$r(t+h) - r(t)$ is close to the tangent of the curve at $r(t)$ as $h \to 0$. But the issue is that $r(t+h) - r(t)$ goes to vector 0.

$\rightsquigarrow$ divide by h to make it not too small:

$$\frac{r(t+h) - r(t)}{h} \longrightarrow r'(t) \ : \text{ tangent vector at } r(t).$$