Lecture 9

Saturday, January 22, 2022 12:18 PM

* Prayer
* Spiritual thought

Length of a curve is $L = \int_a^b |r'(t)|\,dt.$

Ex curve $\begin{cases} x = t \\ y = \frac{4}{3} t^{3/2} \\ z = t^2 \end{cases}$ $\quad 0 \le t \le 1$

Find the length.

$r(t) = \langle t, \frac{4}{3} t^{3/2}, t^2 \rangle$

$r'(t) = \langle 1, \frac{4}{3}\cdot\frac{3}{2} t^{1/2}, 2t \rangle = \langle 1, 2\sqrt{t}, 2t \rangle$

$|r'(t)| = \sqrt{1 + 4t + 4t^2} = \sqrt{(1+2t)^2} = 1+2t$

$L = \int_0^1 (1+2t)\,dt = 2.$

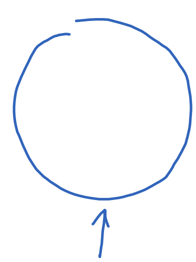
<u>Curvature</u>

how sharp the curve turns.

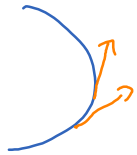
low curvature

high curvature

high curvature



low curvature



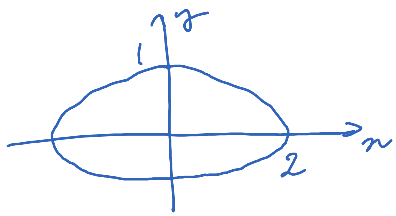
Curvature ~ how quickly the unit tangent vector changes.

$$= \left| \frac{dT}{ds} \right|$$

Equivalent formula:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Ex



$x^2 + 4y^2 = 4$

Find the locations with maximum/minimum curvature.

Parametrize the curve:

$$\left(\frac{x}{2}\right)^2 + y^2 = 1 \rightsquigarrow \begin{cases} \frac{x}{2} = \cos t \\ y = \sin t \\ z = 0 \end{cases} \rightsquigarrow \begin{cases} x = 2\cos t \\ y = \sin t \\ z = 0 \end{cases}$$

Thus, $r(t) = \langle 2\cos t, \sin t, 0 \rangle$

$$r'(t) = \langle -2\sin t, \cos t, 0 \rangle \quad -2\sin t \quad \cos t$$

$$r''(t) = \langle -2\cos t, -\sin t, 0 \rangle \quad -2\cos t \quad -\sin t$$

$$r'(t) \times r''(t) = \langle 0, 0, 2\sin^2 t + 2\cos^2 t \rangle = \langle 0, 0, 2 \rangle$$

$$|r'(t) \times r''(t)| = \sqrt{0^2 + 0^2 + 2^2} = 2$$

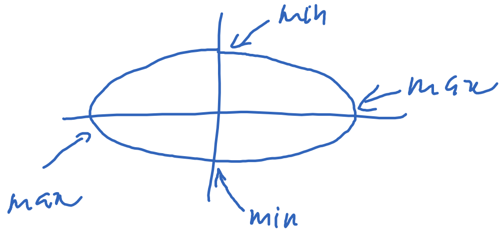
$$|r'(t)| = \sqrt{(-2\sin t)^2 + \cos^2 t + 0^2} = \sqrt{1 + 3\sin^2 t}$$

Curvature:

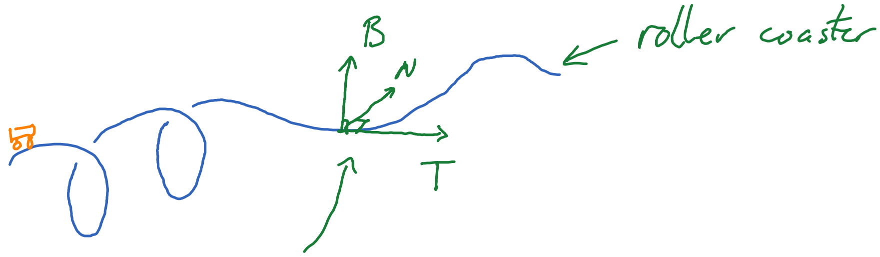
$$k(t) = \frac{2}{(1+3\sin^2 t)^{3/2}}$$

$\rightsquigarrow$ min if $\sin t = 0$

$\rightsquigarrow$ max if $\sin t = \pm 1$



Torsion



moving frame, or Frenet frame, or TNB frame

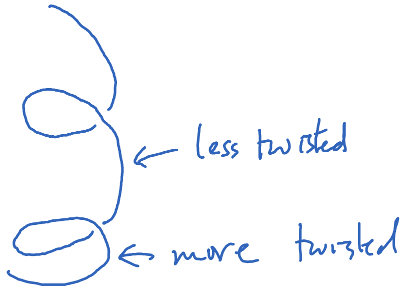
$$N = \frac{T'(t)}{|T'(t)|}, \qquad B = T \times N$$

Torsion = how fast B changes (how twisted the curve is)

$$\frac{dB}{ds} = -\tau N$$

Equivalent formula:

$$\tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}$$



Ex

$$r(t) = \langle \cos t, \sin t, at \rangle$$

$$r'(t) = \langle -\sin t, \cos t, a \rangle$$

$$r''(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$r'''(t) = \langle \sin t, -\cos t, 0 \rangle$$

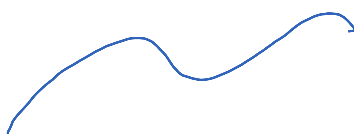
$$r' \times r'' = \langle a\sin t, -a\cos t, 1 \rangle \rightsquigarrow |r' \times r''| = \sqrt{1+a^2}.$$

$$(r' \times r'') \cdot r''' = a\sin^2 t + a\cos^2 t + 0 = a$$

$$\tau = \frac{a}{1+a^2}$$

Note: τ is maximum when $a = 1$.

Motion

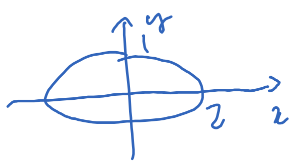


$r(t) =$ position function

$v(t) = r'(t) =$ velocity , $|v(t)| =$ speed

$a(t) = v'(t) = r''(t) =$ acceleration.

Ex



$$\begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$$

where are the places with maximum/minimum speed?

$$r(t) = \langle 2\cos t, \sin t \rangle$$

$$r'(t) = \langle -2\sin t, \cos t \rangle$$

$$|r'(t)| = \sqrt{(-2\sin t)^2 + \cos^2 t} = \sqrt{1 + 3\sin^2 t}$$

min when $\sin t = 0$

max when $\sin t = \pm 1$.