Analysis seminar (11/19/2018)

(NSE):
$$\begin{cases} \partial_t u - \Delta u + u\nabla u + \nabla p = f \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T) \\ u(0) = u_0 \end{cases}$$

* Mild sol.:

$$\partial_t u - \Delta u + \mathbb{P}(u\nabla u) = \mathbb{P} f$$

$$u = \underbrace{e^{\Delta t} u_0}_{U} + \underbrace{\int_0^t e^{\Delta(t-s)} \mathbb{P} f(s)\, ds}_{F} + \underbrace{\int_0^t -e^{-\Delta(t-s)} \mathbb{P}(u\nabla u)\, ds}_{B(u,u)}$$

$X = L^5_{t,x}$. Known: $B: X \times X \to X$ continuous bilinear form.

$$\|u\|_X \le \|U + F\|_X + C\|u\|_X^2$$

$\rightsquigarrow$ unique local solution given by Picard iteration

* Weak sol.

many existing definitions, not equivalent to one another in general due to possible lack of uniqueness.

Leray-Hopf: $u_0 \in L^2$, $f \in L^2_t H^{-1}_x$ $\longrightarrow$ global solution

idea: • regularize the nonlinear term to obtain global and regular solutions

$u\nabla u \longrightarrow (u * \eta_\varepsilon)\nabla u$ (η_ε): mollifiers in $\mathbb{R}^3$ [1]

$u\nabla u \longrightarrow (u * \phi_\varepsilon)\nabla u$ (ϕ_ε): mollifier in $\mathbb{R}^4$ [3]

Galerkin approximation [2], [4, Ch. III, §3]

• Use the energy identity

$$\int_{\mathbb{R}^3} \frac{1}{2}|u|^2 dx + \int_0^t\int_{\mathbb{R}^3} |\nabla u|^2 dx ds \overset{(\le)}{=} \int_{\mathbb{R}^3} \frac{1}{2}|u_0|^2 dx$$

to pass to the limit as $\varepsilon \to 0$.

Lémarie-Rieusset solution:

$u_0 \in L^2_{uloc}$, $f = 0$ $\longrightarrow$ local weak solution

Scheffer (1977) introduced local energy solutions:

$$\int_{\mathbb{R}^3} |\nabla u|^2 \phi\, dx \le \int_0^t\int_{\mathbb{R}^3} \frac{1}{2}|u|^2(\partial_t\phi + \Delta\phi) + \left(\frac{1}{2}|u|^2 + p\right) u\nabla\phi + fu\phi\, dx ds$$

This allows $u_0 \in L^2_{uloc}$. Scheffer showed global existence when $u_0 \in L^2$ and $f=0$.

C-K-N [3] generalized to $u_0 \in L^2$, $f \in L^2_t H^{-1}_x$.

Lemarie-Rieusset [5]:

$u^{(\varepsilon)}$ ... solution to $(NSE)_\varepsilon$: $u\nabla u \to (u * \eta_\varepsilon)\nabla u$

$$\|u^{(\varepsilon)}(t)\|_{L^2_{uloc}} \leq \frac{\|u_0\|^2_{L^2_{uloc}}}{(1-Ct)^{1/4}}$$

take $\varepsilon \to 0$: local weak solution to (NSE)

Possible blowup of energy (of $u^{(\varepsilon)}$) after finite time is due to energy coming from infinity: $u_0 \in L^2_{uloc}$, so it's possible that $\int_{B_1(x_0)} \frac{1}{2}|u_0|^2 dx$ is large when $x_0 \to \infty$.

If $u_0 \in E$ .... closure of C_c^∞ in L^2_{uloc} then global weak solutions exist.

The construction of $u^{(\varepsilon)}$ and the limit process rely on the fact that the pressure is essentially eliminated from the scheme:

$$\Delta p = -\operatorname{div}(u\nabla u)$$

$$\Rightarrow \quad p = R_j R_k (u_j u_k)$$: pressure only comes from convection.

where $\widehat{R_j(f)} = \frac{\xi_j}{i|\xi|}\hat{f}$ is Riesz transform.

How about domain with boundary?

Consider the Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \\ u|_{\partial \mathbb{R}^3_+} = 0 \\ u(0) = 0 \end{cases}$$

Helmholtz decomposition: $v = \underbrace{w}_{\mathbb{P}v} + \nabla\phi$ in $\mathbb{R}^3_+$

where $w \cdot n = 0$, $\operatorname{div} w = 0$

$\mathbb{P}\Delta u \neq \Delta u$ because $\Delta u \cdot n = -\frac{\partial^2 u_3}{\partial x_3^2} \neq 0$

In fact, $p = \underset{\sim}{p_d} + p_c + \underset{\sim}{p_f}$

p_d: from diffusion at the boundary
p_c: pressure coming from convection term
p_f: from external force

$$\begin{cases} \Delta p_d = 0 \\ \dfrac{\partial p_d}{\partial x_3}\Big|_{x_3=0} = -\dfrac{\partial^2 u_3}{\partial x_3^2}\Big|_{x_3=0} \end{cases} \rightsquigarrow \text{the pressure required to preserve non-slip boundary condition.}$$

The difficulty of repeating L-S's argument to obtain global sol. is caused by this pressure term (called harmonic / boundary pressure).

One needs a different way to decompose the pressure. This leads to the need of new way to decompose the solution.

This is done by Prange et al. [8] using a decomposition technique (sw-solutions).

Sw-solutions (Stokes / strong / split - weak), Seregin-Sverak 2017

$u = v + w$, $p = q + \pi$, $\Omega = \mathbb{R}^3_+$ or $\mathbb{R}^3$

here the harm. pressure is split into two, one for each system

$$\begin{cases} \partial_t v - \Delta v + \nabla q = f \\ \operatorname{div} v = 0 \\ v|_{\partial\Omega} = 0 \\ v(0) = u_0 \end{cases} \qquad \begin{cases} \partial_t w - \Delta w + \nabla \pi = -u \cdot \nabla u \\ \operatorname{div} w = 0 \\ w|_{\partial\Omega} = 0 \\ w(0) = 0 \end{cases}$$

→ The problem of pressure decomposition in NSE becomes the problem of estimating the pressure in SE, which is a linear eq.

Features:

with $f \in L^2_{loc}$, $v \in (L^\infty_t L^2_x \cap L^2_t H^1_x)_{loc}$

v satisfies
- local energy identity
- maximal regularity
- regular, global

w satisfies energy inequality (full scale in x)

This decomposition helps simplify a number of proofs:

[Thm: (a priori local energy estimate)

$$\|u\|_{L^\infty_t L^2_x \cap L^2_t H^1_x (K\times(0,T))} \lesssim \Phi\left(\|f\|_{L^{5/3}_{t,x}}, \|u_0\|_{L^3}\right)$$]

v .... ok due to linear SE

w:
$$\int_\Omega \frac{1}{2}|w|^2 + \int_0^t \int_\Omega |\nabla w|^2 \,dx\,ds = \underbrace{\int_0^t \int_\Omega v\otimes u : \nabla w \,dx\,ds}_{= A+B}$$

$$A = \int_0^t \int_\Omega v\otimes v : \nabla w \,dx\,ds \le \int_0^t \|v\|_{L^4}^2 \|\nabla w\|_{L^2}\,ds$$

$$\le \int_0^t \Big(\underbrace{4\|v\|_{L^4}^4}_{} + \frac{1}{2}\|\nabla w\|_{L^2}^2\Big)\,ds$$

$$\le 4\sqrt{t}\,\|v\|^4_{L^8_t L^4_x}$$

$$\lesssim \sqrt{t}\left(\|f\|_{L^{5/3}_{t,x}} + \|u_0\|_{L^3}\right)$$

$$B = \int_0^t \int_\Omega v\otimes w : \nabla w \,dx\,ds \le \int_0^t \|v\|_{L^5} \|w\|_{L^2}^{2/5} \|\nabla w\|_{L^2}^{8/5}\,ds$$

$$\le \int_0^t 4\|v\|_{L^5}^5 \|w\|_{L^2}^2\,ds + \frac{1}{4}\int_0^t \|\nabla w\|_{L^2}^2\,ds$$

Then
$$A+B \le C_{f,u_0,T} + \frac{1}{2}\int_0^t \int_\Omega |\nabla w|^2\,dx\,ds + \int_0^t 4\|v\|_{L^5}^5 \|w\|_{L^2}^2\,ds$$

Gronwall inequality: $\|w\|_{L^\infty_t L^2_x} \lesssim C_{f,u_0,T}$

$$\|\nabla w\|_{L^2_t L^2_x} \lesssim C_{f,u_0,T}$$

* The key to study boundary reg. is to estimate pressure in local scale.
* How to estimate the pressure from the SE?

$$\partial_t u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0) = 0.$$

Maximal regularity: $X = L^r_t L^q_x$, $1 < r, q < \infty$

$$\|u\|_X + \|\partial_t u\|_X + \|\nabla u\|_X + \|\nabla^2 u\|_X + \|\nabla p\|_X \lesssim \|f\|_X$$

Why is this true?

When $\Omega = \mathbb{R}^3$, this is maximal regularity of the heat eq.

(since ∇p only comes from f through Helmholtz decomposition)

$$\partial_t u - \Delta u = f \qquad (1)$$

The case $r = q = 2$ is rather simple: square both sides

$$\int |\partial_t u|^2 - \underbrace{\int 2\partial_t u \cdot \Delta u \, dx}_{= -\frac{d}{dt}\int |\nabla u|^2 dx} + \int |\Delta u|^2 = \int f^2$$

$$\Rightarrow \iint (|\partial_t u|^2 + |\Delta u|^2) \leq \iint f^2 \, dx \, ds$$

To get an estimate for u and ∇u, multiply both sides of (1) by u:

$$\int |u|^2 + \iint |\nabla u|^2 dx\, ds \leq \iint f^2 \, dx\, ds$$

For $r = q$: one needs the theory of singular integral / Fourier multi.

Fourier transform in x: $\partial_t \bar{u} + |\xi|^2 \bar{u} = \bar{f}$

Take Fourier transform in t: $i\xi_0 \hat{u} + |\xi|^2 \hat{u} = \hat{f}$

$$\Rightarrow \hat{u} = \frac{1}{|\xi|^2 + i\xi_0} \hat{f}$$

$$\Rightarrow \widehat{\nabla^2 u} = \underbrace{\frac{-\xi \otimes \xi}{|\xi|^2 + i\xi_0}}_{m(\xi, \xi_0) \text{ - Fourier multiplier}} \hat{f}$$

Easy to check: $|\partial^\alpha m(\xi, \xi_0)| \leq C |(\xi, \xi_0)|^{-\alpha}$

By Hörmander–Mikhlin multiplier thm: the map $f \mapsto \nabla^2 u$ is bounded from $L^q_{t,x}$ to $L^q_{t,x}$ for all $1 < q < \infty$.

A similar procedure is carried out for $\Omega = \mathbb{R}^3_+$ (see [6], [7, thm 3.1])

(The case $r = q = 2$ is still simple: square both sides of the SE,

$$\iint (|\partial_t u|^2 + |\Delta u|^2 + |\nabla p|^2) \lesssim \int f^2 \;)$$

The kernel of Stokes eqs. in half space: [11]

$$u(x,t) = \int_{\mathbb{R}^3_+} G(x,y,t)\, u_0(y)\, dy$$

$$= \Gamma(t) * e_d(u_0) + \left(\chi_{x_3>0}\, \partial_3^2 \Phi\right) * \Gamma(\cdot,t) * J e_0(u_0)$$

$\left(J = \begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix} \right)$

- $e_d(u_0)$: odd extension
- Φ: fundamental sol. of Laplace eq. in $\mathbb{R}^3$.
- $e_0(u_0)$: extension by 0

$\Gamma(t) * e_d(u_0)$: sol. to heat eq. in $\mathbb{R}^3_+$, obtained by odd reflection

$\left(\chi_{x_3>0}\, \partial_3^2 \Phi\right) * \Gamma(\cdot,t) * J e_0(u_0)$: Correction term to ensure divergence free condition

* Persistence of singularity near the boundary:

$$f^k \to f \quad , \quad \|f^k\|_Y \downarrow \|f\|$$

f^k yields blow up sol. (u^k, p^k); f yields mild sol. (u, p)

ε- regularity:

$$\frac{1}{r^2} \int_{Q_r^+} \left(|u^k|^3 + |p^k|^{3/2}\right) dx\, dt \geq \varepsilon \qquad \forall k, \forall r$$

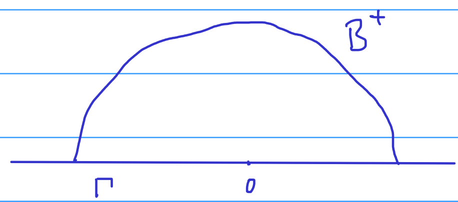
Want: $\displaystyle \frac{1}{r^2} \int_{Q_r^+} |p^k - p|^{3/2}\, dx\, dt \to 0 \quad \text{as } k \to \infty,\ r \to 0$

We need a special decomposition of the pressure. Each component needs to have either strong convergence in $L^{3/2}_{t,x}$ or higher integrability. (e.g. $p \in L^{3/2}_t L^9_x$)

* Estimate the pressure near the boundary:

the system of w is further split into 2 systems:

localized in space
$$\begin{cases} \partial_t w^{(1)} - \Delta w^{(1)} + \nabla \pi^{(1)} = -u \cdot \nabla u \\ \operatorname{div} w^{(1)} = 0 \\ w^{(1)}|_{\partial B^+} = 0 \\ w^{(1)}(0) = 0 \end{cases}
\qquad
\begin{cases} \partial_t w^{(2)} - \Delta w^{(2)} + \nabla \pi^{(2)} = 0 \\ \operatorname{div} w^{(2)} = 0 \\ w^{(2)}|_{\Gamma} = 0 \\ w^{(2)}(0) = 0 \end{cases}$$



Maximal regularity : $\|\nabla \pi^{(1)}\|_{L^r_t L^q_x} \lesssim \| u \nabla u\|_{L^r_t L^q_x}$

↖ pressure coming from convection term
This is not a new term

$\nabla \pi^{(2)}$: has improved reg. in interior domain.

Seregin [16] shows that

$$\left[\begin{aligned} &\|w^{(2)}\|_Y + \|\partial_t w^{(2)}\|_Y + \|\nabla w^{(2)}\|_Y + \|\nabla^2 w^{(2)}\|_Y + \|\nabla \pi^{(2)}\|_Y \\ &\qquad \lesssim \|w^{(2)}\|_X + \|\nabla w^{(2)}\|_X + \|\pi^{(2)}\|_X \\ &\text{for } \; Y = L^r_t L^{q_2}_x , \; X = L^r_t L^{q_1}_x , \quad q_1 < q_2 \end{aligned} \right]$$

$\pi^{(2)}$ is the only new pressure term (instead of harmonic pressure), and is far easier to treat compare to the harmonic pressure.

Why should this be true? Examine the heat equation in $B^+ \times (0,1)$.

$$\partial_t u - \Delta u = 0$$

Cutoff u to localize the problem in $B^+ \times (0,1)$ with homogeneous boundary condition.

$$u \longrightarrow u\chi = v$$

$$\partial_t v - \Delta v = \underbrace{v\, \partial_t \chi - \nabla u \nabla \chi - u \Delta \chi}_{\in L^r_t L^{q_1}_x}$$

Then by maximal regularity, $\partial_t v, \nabla v, v, \nabla v \in L^r_t L^{q_1}_x$.

Then by parabolic embedding theorems, $v \in L^r_t L^{q_2}_x$.

Similar arguments apply for the Stokes equations. Only note that the cutoff should be Bogovskii cutoff, i.e.

$$v = u\chi + \underbrace{\phi}_{\text{correction term}}$$

to make sure that $\operatorname{div} v = 0$

References:

[1] Leray 1934

[2] Hopf 1951

[3] Caffarelli - Kohn - Nirenberg 1982

[4] Temam, "Theory of Navier-Stokes equations" 1975

[5] Lémarie-Rieusset, "Recent developments of the NSE" 2002

[6] McCracken, "Resolvent problem for the NSE on the half space" 1981

[7] Solonnikov, "Estimates of the solutions of the nonstationary Navier-Stokes system", 1973

[8] Maekawa-Miura-Prange "Local energy weak sol. in half space" 2018

[9] Seregin-Sverak, "On global weak sol... ", 2017.

[10] Seregin, "Some estimates near the boundary..." 2003.

[11] Tuan Pham's PhD thesis, U. of Minnesota 2018.