On the blowup, nonuniqueness, and stochastic explosion of PDE

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Diffusion equation – Probabilistic representation

In $\mathbb{R}^d \times (0, \infty)$, consider the initial-value problem

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u & = & 0, \\ u(x,0) & = & u_0(x). \end{cases}$$

Classical solution:

$$u(x,t) = \int_{\mathbb{R}^d} \underbrace{\frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|y-x|^2}{2t}\right)}_{\Phi(y-x,t)} u_0(y)dy.$$

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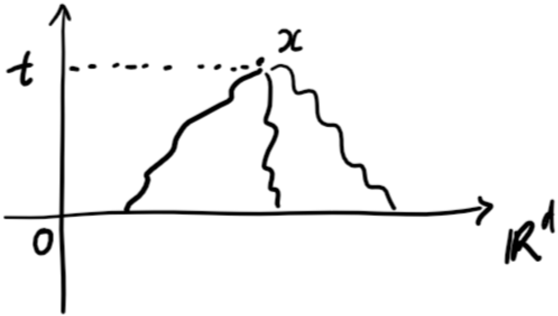
Observe: $\Phi(\cdot - x, t)$ is the p.d.f of an $\mathcal{N}(x, t\, I_d)$-random variable in $\mathbb{R}^d$, e.g. Brownian motion B_t^x.

$$u(x,t) = \mathbb{E}[u_0(B_t^x)].$$

Diffusion equation – Probabilistic representation



"Stochastic characteristic curves"

Reaction-Diffusion equation – Probabilistic representation

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u &= -K(x)u, \\ u(x,0) &= u_0(x). \end{cases}$$

Feynman-Kac formula (1940s):

$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\exp\left(-\int_0^t K(B_s^x)ds\right)\right].$$

The problem can be formulated and generalized (with drift term ∇u and forcing f) by Itô calculus (1950s).

KPP-Fisher equation

In $\mathbb{R} \times (0, \infty)$, consider the equation (Kolmogorov-Petrovskii-Piskunov (KPP), Fisher, 1937):

$$\begin{cases} u_t - \frac{1}{2} u_{xx} &= u^2 - u, \\ u(x,0) &= u_0(x). \end{cases}$$

With $\Psi = e^{-t}\Phi$,

$$u(x,t) = \int_{\mathbb{R}} \Psi(x-y,t) u_0(y) dy + \int_0^t \int_{\mathbb{R}} \Psi(x-y,s) u^2(y,t-s) dy ds.$$

Noting that Ψ is a p.d.f on $\mathbb{R} \times (0, \infty)$, McKean (1975) gave a probabilistic description of this equation by branching process.

KPP-Fisher equation

$T \sim Exp(1)$: holding time (the clock).

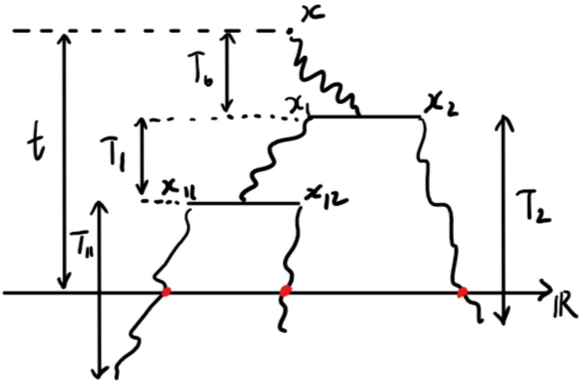
$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\mathbb{1}_{T>t}\right] + \mathbb{E}\left[u^2(B_T^x, t-T)\mathbb{1}_{T\le t}\right]$$

In other words, $u(x,t) = \mathbb{E}[\mathbf{X}(x,t)]$ where

$$\mathbf{X}(x,t) = \begin{cases} u_0(B_t^x) & \text{if} \quad T > t, \\ \mathbf{X}^{(1)}(B_T^x, t-T)\,\mathbf{X}^{(2)}(B_T^x, t-T) & \text{if} \quad T \le t. \end{cases}$$

Here $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are i.i.d copies of $\mathbf{X}$ and are independent of T.

KPP-Fisher equation – Branching process



$$\mathbf{X}(x,t) = u_0(B^{x_{11}}_{t-T_0-T_1})u_0(B^{x_{12}}_{t-T_0-T_1})u_0(B^{x_2}_{t-T_0}).$$

Diffusion-reaction equation – Fourier domain – Ex. 1

The heat operator $\partial_t - \Delta$ naturally induces a clock in the Fourier domain. For example,

$$u_t - u_{xx} = bu, \quad u(x,0) = u_0(x).$$

In Fourier domain,

$$\hat{u}(\xi,t) = e^{-t\xi^2}\hat{u}_0(\xi) + \int_0^t e^{-s\xi^2} b\hat{u}(\xi,t-s)ds.$$

Put $\chi = \xi^2\hat{u}$. Then

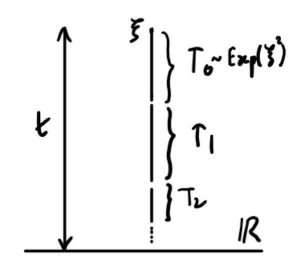
$$\chi(\xi,t) = e^{-t\xi^2}\chi_0(\xi) + \int_0^t \underbrace{\xi^2 e^{-s\xi^2}}_{\text{p.d.f}} \frac{b}{\xi^2}\chi(\xi,t-s)ds.$$

Diffusion-reaction equation – Fourier domain – Ex. 1

$$\chi(\xi, t) = \mathbb{E}[\mathbf{X}(\xi, t)]$$

where

$$\mathbf{X}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T > t, \\ \frac{b}{\xi^2}\mathbf{X}(\xi, t - T) & \text{if} \quad T \leq t. \end{cases}$$



$$\mathbf{X}(\xi, t) = \left(\frac{b}{\xi^2}\right)^{N_t} \chi_0(\xi), \quad N_t = \inf\{n : T_0 + T_1 + ... + T_n > t\}$$

Diffusion-reaction equation – Fourier domain – Ex. 2

$$u_t - u_{xx} = (\cos x)u, \quad u(x,0) = u_0(x).$$

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-t\xi^2} + \frac{c}{2}\int_0^t \xi^2 e^{-s\xi^2}\left(\frac{\hat{u}(\xi-1,t-s)}{\xi^2} + \frac{\hat{u}(\xi+1,t-s)}{\xi^2}\right)ds$$

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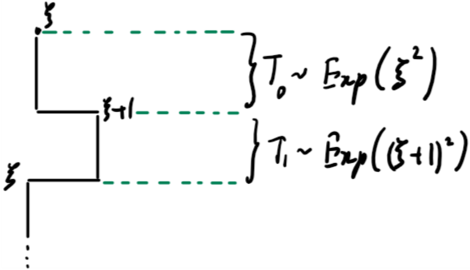


$$\mathbf{X}(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T > t, \\ \frac{c}{\xi^2}\mathbf{X}(W, t-T) & \text{if} \quad T \leq t. \end{cases}$$

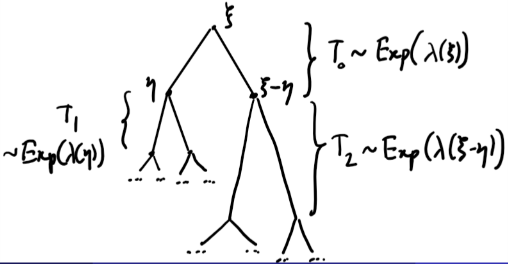
$$\mathbb{P}_\xi(W = \xi - 1) = \mathbb{P}_\xi(W = \xi + 1) = 1/2.$$

$$u_t - \frac{1}{2}u_{xx} = u^2 - u, \quad u(x,0) = u_0(x).$$

Fourier domain:

$$\hat{u}_t + \underbrace{\left(1 + \frac{1}{2}|\xi|^2\right)}_{\lambda(\xi)} \hat{u} = \widehat{u^2}.$$

$$\hat{u}(\xi,t) = e^{-\lambda(\xi)t}\hat{u}_0 + c\int_0^t e^{-\lambda(\xi)s}\hat{u}(\eta,t-s)\hat{u}(\xi-\eta,t-s)d\eta ds$$



Navier-Stokes equations

$$(\mathrm{NSE}): \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ \mathrm{div}\ u = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}$$

Integro-differential equation:

$$u(x,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta s} \mathbf{P} \mathrm{div}[u(t-s) \otimes u(t-s)] ds.$$

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In Fourier domain:

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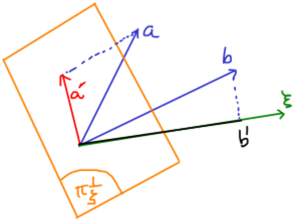
\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + c_0\int_0^t e^{-|\xi|^2 s}|\xi|\int_{\mathbb{R}^d}\hat{u}(\eta,t-s)\odot_\xi \hat{u}(\xi-\eta,t-s)d\eta ds
$$

where $a\odot_\xi b = -i(e_\xi\cdot b)(\pi_{\xi^\perp}a)$.

Fourier-transformed Navier-Stokes equations (FNS)



$a \odot_\xi b = -i\, b' a'$

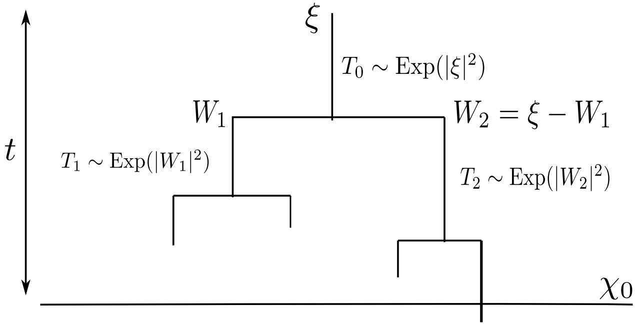
Normalization to (FNS): LJS 1997, Bhattacharya et al 2003.

$$\begin{aligned}\chi(\xi, t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi) d\eta ds\end{aligned}$$

where $\chi = c_0 \hat{u}/h$ and $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$.

h: *majorizing kernel*, i.e. $h * h = |\xi| h$.

Cascade structure of FNS

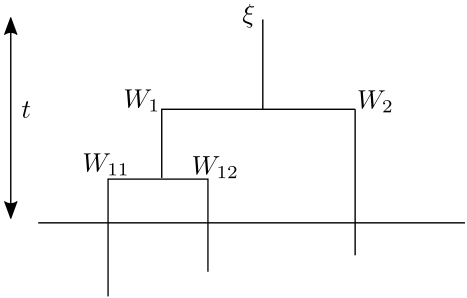


Define a stochastic multiplicative functional recursively as

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\mathrm{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\mathrm{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \le t. \end{cases}$$

An example of $\mathbf{X}_{\mathrm{FNS}}$

Consider the following event:



On this event,

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_{\xi} \chi_0(W_2).$$

Three ingredients: clocks, branching process, product.
Cascade structure = clocks + branching process.

Stochastic cascade solutions – Two issues

We referred to solutions given by the expectation of a multiplicative stochastic functional $\mathbf{X}$ as *stochastic cascade solutions*.

Cascade structure = clocks + branching process.

There are two potential issues with this construction:

- Stochastic explosion: the branching process may never stop, making $\mathbf{X}$ not well-defined.
- Existence of the expectation: it may happen that $\mathbb{E}|\mathbf{X}| = \infty$.

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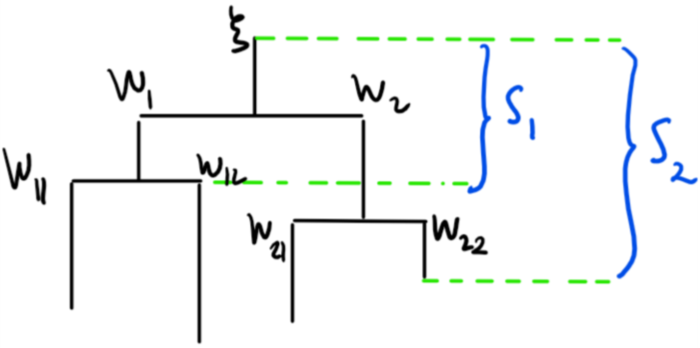
Existence of expectation $\longleftrightarrow$ smallness of initial condition.

Stochastic explosion: (1) *Can it happen?* (2) *Any connection with non-uniqueness?*

Explosion

$$S_n = \min_{|\nu|=n} \sum_{j=0}^{n} T_{\nu|j}, \qquad S = \lim_{n\to\infty} S_n = \sup_{n\in\mathbb{N}} S_n$$

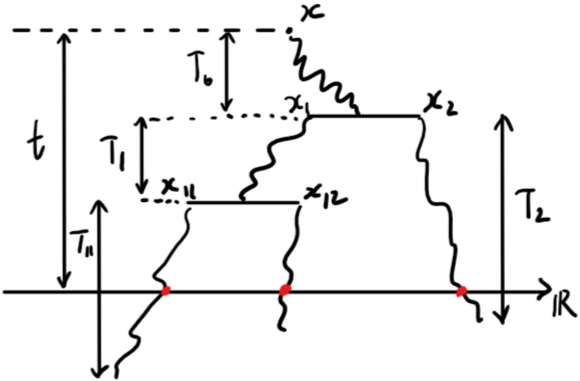


Explosion event: $\{S < \infty\}$.

Non-explosion event : $\{S = \infty\}$.

Examples of non-explosion

$$u_t - u_{xx} = u^2 - u, \quad u(x,0) = u_0(x).$$



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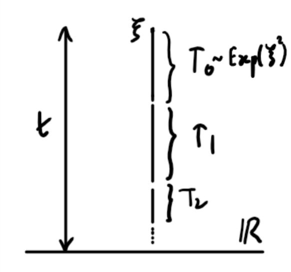

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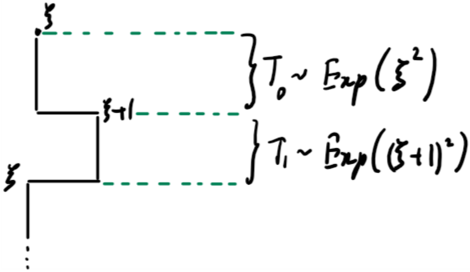

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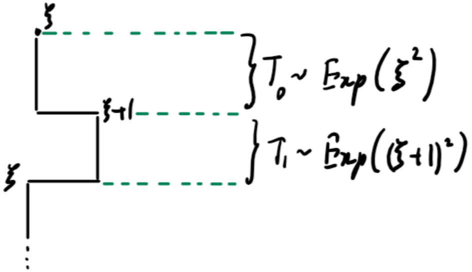

$$\mathbb{P}_\xi(W = \xi - 1) = \mathbb{P}_\xi(W = \xi + 1) = 1/2.$$

$$\sum_{n=1}^{\infty} T_{\nu|n} = \sum_{n=1}^{\infty} \frac{\bar{T}_{\nu|n}}{|W_{\nu|n}|^2} \geq \sum_{k=1}^{\infty} \frac{\bar{T}_{\nu_k}}{\xi^2} = \frac{1}{\xi^2} \sum_{k=1}^{\infty} \bar{T}_{\nu_k} = \infty \quad a.s.$$

Explosion

Branching process may never stop, potentially making $\mathbf{X}_{FNS}$ not well-defined.

- Property of cascade structure, not of product.
- Depending on the majorizing kernel h.
- 3D self-similar cascade $h_{\mathrm{dilog}}(\xi) = C|\xi|^{-2}$: stochastic explosion a.s. (Dascaliuc, Pham, Thomann, Waymire 2019)
- 3D Bessel cascade $h_{\mathrm{b}}(\xi) = C|\xi|^{-1}e^{-|\xi|}$: non-explosive a.s. (Orum, Pham 2019)

Stochastic explosion of NSE

Explosion happens a.s. for the self-similar cascade, how can we define a stochastic cascade solution and is it unique?

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Introducing a *ground state*:

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Ignore the product and regard $\mathbf{X}_n$ as a scalar:

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Nonuniqueness (of cheap NSE)

- For $\chi_0 \equiv 1$ and $\mathbf{X}_0 \equiv 1$ then $\mathbf{X}_n = 1$ for all n. Thus, $\chi = \lim \mathbb{E}\mathbf{X}_n = 1$.
- For $\chi_0 \equiv 1$ and $\mathbf{X}_0 \equiv 0$ then $\chi = \lim \mathbb{E}\mathbf{X}_n = \mathbb{P}(S_\xi > t)$. where S_ξ is the shortest branch of the tree rooted at ξ.
- Both of functions χ above solve the equation

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta,t-s)\chi(\xi-\eta,t-s)H(\eta|\xi)d\eta ds\end{aligned}$$

- In physical domain: $\partial_t u - \Delta u = \sqrt{-\Delta}(u^2)$ and $u(x,0) = c/|x|$, called the *cheap NSE* (Montgomery-Smith 2002).

Another toy model: α-Riccati equation

$$u' + u = u^2(\alpha t), \quad u(0) = u_0.$$

$$\mathbf{X}(t) = \begin{cases} u_0 & \text{if} \quad T > t, \\ \mathbf{X}^{(1)}(\alpha(t-T))\mathbf{X}^{(2)}(\alpha(t-T)) & \text{if} \quad T \leq t. \end{cases}$$

$$\mathbb{E}S \leq \mathbb{E}\left(T_0 + \frac{T_1}{\alpha} + \frac{T_{11}}{\alpha^2} + \frac{T_{111}}{\alpha^3} + \cdots\right) = \frac{\alpha}{\alpha - 1} < \infty$$

Thank You!