Research Statement

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My research interest is in Partial Differential Equations, Numerical Analysis, Probability theory, and their applications in Fluid Mechanics and Mathematical Biology. I am also interested in the relation between stochastic processes and the analysis of PDE. My recent research has been on the regularity theory of the Navier-Stokes Equations and several modeling equations. In the following, I will briefly describe some of my recent projects, their motivations, and my future research plans.

1 Global regularity criterion based on approximate solutions

Let us consider the Cauchy problem for the incompressible Navier-Stokes equations:

$$\text{(NSE)} : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

The system has a scaling property:

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t), \quad p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t), \quad u_0(x) \to \lambda u_0(\lambda x)$$

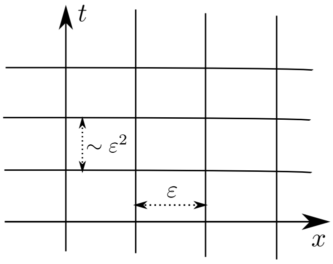
where $\lambda \in \mathbb{R}$. While the global-wellposedness of (NSE) is still not known, a variety of regularized systems obtained by mollifying the nonlinear term $u \cdot \nabla u$ are known to be globally wellposed. Regularizations of the nonlinear term often involve a resolution parameter ε. Two well-known examples are

1. The classical Leray regularization $(u * \eta_\varepsilon) \cdot \nabla u$,
2. The regularization $P_\varepsilon(u \cdot \nabla u)$, where P_ε is an orthogonal projection on $L^2(\mathbb{R}^3)$ whose Fourier multiplier is a smooth cutoff function supported in the ball $B_{2\varepsilon^{-1}}$ and equal to 1 in the ball $B_{\varepsilon^{-1}}$.

These approximations generate a family of global smooth approximate solutions to (NSE), which can be useful for the construction of global weak solutions. Full information on the behavior of a sequence of approximate solutions as $\varepsilon \downarrow 0$ gives information on the exact solution. However, in practice we only have information about finitely many approximate solutions. Let us in fact assume that we know only one approximate solution for a certain value of ε. It is natural to consider two following questions:

> (Q) Under what conditions can we infer global regularity for the exact solution? How large can ε be in terms of the "observable" quantity M to still guarantee global regularity of exact solutions?

These questions have been addressed by Li in [14] from a computational perspective. He considered a discretized Navier-Stokes system on a polyhedron and showed that if a numerical solution u_ε corresponding to some mesh size ε is of size M (the L^∞-norm) with $\varepsilon \sim \exp(-(\|u_0\|_{H_0^1\cap H^2}+1)^\alpha M^\alpha)$, where α is a large number, then the exact solution is regular for all times. This leads to a type of global regularity criteria based on approximate solutions. It differs from most of the well-known criteria in that they often require some smallness condition on the initial condition, e.g. [10, 11], or some symmetry on the initial condition, e.g. [9, 12].



My paper [19] is an investigation of a global regularity criterion based on continuous approximate solutions on the whole space $\mathbb{R}^3$. In this setting, the problem already contains the key difficulties but is technically simpler. Consider the regularized Navier-Stokes system:

$$(\mathrm{NSE})_\varepsilon : \quad \begin{cases} \partial_t u - \Delta u + [u\cdot\nabla u]_\varepsilon + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(\cdot,0) = u_0 \end{cases}$$

where $[u\cdot\nabla u]_\varepsilon$ is used as a common notation for two types of regularizations mentioned above. To answer the first question, I give a simple criterion involving the resolution parameter ε, the size M (the L^∞-norm) of the corresponding approximate solution u_ε, and the total energy $\|u_0\|_{L^2}^2$ which guarantees that the exact solution is regular globally. The scaling symmetry can be used to predict possible answers to the second question. Often, the resolution parameter ε can be normalized to have the same scaling as spatial length. Since both ε and M^{-1} are of dimension length, it seems reasonable to expect a rate of $\varepsilon \sim M^{-1}$. However, the time-dependence nature complicates the problem. For one reason, the initial energy $\|u_0\|_{L^2}^2$ also has the same scaling as spatial length and, thus, can be considered as another length scale of the problem. Another reason is that the higher initial energy naturally requires finer resolution in order to capture complex structures of the exact solution at small scales. I obtain the following result:

Theorem 1 ([19]) *Suppose (NSE)$_\varepsilon$ has a global mild solution u_ε bounded by M. If $\varepsilon \sim M^{-1}\exp(-\|u_0\|_{L^2}^4 M^2)$ then (NSE) has a global mild solution bounded by $2M$.*

This is an improvement of the condition in [14] although our regularizations are different and, in particular, infinite-dimensional. I give two different approaches, global estimates and local estimates, leading to essentially the same result. The main difficulty is to keep track of the error in the solution as it propagates over time. The main strategy is to estimate the growth of local energy over each time-step of order $O(M^{-2})$, and then use a generalized ϵ-regularity criterion to show local regularity.

2 Le Jan–Sznitman cascade solutions

For $d \geq 1$, let us consider the Cauchy problem for the d-dimensional incompressible Navier-Stokes equations:

$$\text{(NSE)}: \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^d. \end{cases}$$

The system is invariant under the natural scalings $u_0(x) \to \lambda u_0(\lambda x)$ and $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$. The well-posedness of (NSE) is not known for any $d \geq 3$, although different partial results are known depending on what class of solutions is under consideration.

The symbiotic relation between stochastic processes and the analysis of PDE has a long and rich history illustrated, for example, by the role of Brownian motion and general diffusions in the study of properties of harmonic functions and parabolic PDE. The seminal work [16] of McKean in 1975 is perhaps the first example of the use of branching processes in the analysis of semilinear parabolic equations. In 1997, Le Jan and Sznitman used a similar method for the 3D Navier-Stokes equations [13]. They obtained a solution to the integral equation of the Fourier transformed Navier-Stokes equations (FNS). This class of solutions is known as *cascade solutions*. The well-posedness of (FNS) in the class of cascade solutions is still open.

Le Jan and Sznitman's construction of cascade solutions requires overcoming two obstacles, the understanding of which can shed light on the existence and uniqueness of cascade solutions, and the connection between cascade solutions and mild solutions (i.e. solutions obtained by fixed point method) of (FNS). I now briefly describe these obstacles. Then I will describe three of my research projects on dealing with them. To construct a cascade solution, one normalizes $\hat{u}$ to $\chi = c\hat{u}/h$. Here $h = h(\xi) > 0$, called *majorizing kernel*, is a function such that $H(\eta|\xi) = h(\eta)h(\xi-\eta)/(|\xi|h(\xi))$ is a probability density with respect to η. Then χ satisfies

$$\text{(FNS)}: \ \chi(\xi,t) = e^{-t|\xi|^2}\chi_0(\xi) + \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi) d\eta ds.$$

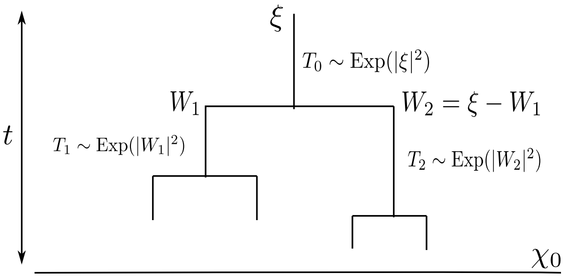
For each $\xi \in \mathbb{R}^d \backslash \{0\}$ and $t > 0$, one defines a stochastic functional $\mathbf{X}$ recursively as

$$\mathbf{X}(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if } T_0 > t, \\ \mathbf{X}^{(1)}(W_1, t-T_0) \odot_\xi \mathbf{X}^{(2)}(\xi - W_1, t - T_0) & \text{if } T_0 \leq t \end{cases}$$

where

- T_0 is an exponentially distributed random variable with mean $|\xi|^{-2}$,
- W_1 is an $\mathbb{R}^d$-random variable independent of T_0 with conditional distribution $H(\eta|\xi)$,
- $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of $\mathbf{X}$.

The definition of $\mathbf{X}$ needs three ingredients: time clocks, a branching process, and a product $\odot_\xi$. The product satisfies $|a \odot_\xi b| \leq |a||b|$ but is neither commutative nor associative. The branching process, as illustrated in the figure below, can be described intuitively as follows: on the full binary tree, label the root by ξ. After a waiting time T_0, we either stop the branching process (when $T_0 > t$) or split into two branches (when $T_0 \leq t$). In the latter scenario, a wave number W_1 is sampled according to distribution $H(\eta|\xi)$. Then the root of the first branch is labeled by W_1, the root of the second branch by $\xi - W_1$. The process continues at each branch independently of each other. The combination of the clocks and branching process is called as *cascade structure*. In closed form, $\mathbf{X}$ is a product of the values of χ_0 at many different



locations. Modulo a few technical modifications, *cascade solution* to (FNS) is essentially defined as $\chi(\xi,t) = \mathbb{E}_{\xi,t}\mathbf{X}$. There are two major issues underlying this definition.

The first issue is *stochastic explosion*: the branching process might keep going indefinitely, potentially making the stochastic functional $\mathbf{X}$ not well-defined. Le Jan and Sznitman overcame this obstacle by a somewhat artificial procedure: they incorporated a force term into the definition of $\mathbf{X}$, even in the case of no forcing, as a means to terminate the cascade. Stochastic explosion is an intrinsic property of the cascade structure described above and depends only on the choice of majorizing kernel h. Knowing whether explosion can happen for a given h is, on one hand, a natural question from probabilistic perspective. On the other hand, from PDE perspective, explosion is equivalent to nonuniqueness of a scalar pseudo-differential equation

$$\text{(gNSE):}\quad \begin{cases} \partial_t u - \Delta u &= \sqrt{-\Delta}(u^2) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ u(\cdot,0) &= c_1 \check{h} \qquad\quad\;\; \text{in} \quad \mathbb{R}^d. \end{cases}$$

known as the *genealogical Navier-Stokes equation.* Among the known majorizing kernels in $\mathbb{R}^3$ are the *self-similar* kernel $h_s(\xi) = c|\xi|^{-2}$ and the *Bessel* kernel $h_b(\xi) = c|\xi|^{-1}e^{-|\xi|}$. In my paper [5] with Dascaliuc, Thomann and Waymire, we show the following:

Theorem 2 ([5]) *The self-similar cascade of the 3D Navier-Stokes equations is explosive.*

As a consequence, (gNSE) has two solutions in the class $\{u : \; |\xi|^2|\hat{u}| \le 1\}$. A distinct difference between the self-similar kernel and the Bessel kernel is that the latter is less singular at 0 and has rapid decay at infinity. Heuristically, the corresponding distribution $H(\eta|\xi)$ puts more weight on smaller wave numbers, thereby discouraging explosion. Although it is not yet clear how to show non-explosion by probability techniques, the PDE picture is clearer. In my upcoming paper [18] with Chris Orum, we show the following:

Theorem 3 ([18]) *The Bessel cascade of the 3D Navier-Stokes equations is non-explosive.*

The key observation is that the Bessel kernel puts the initial condition $u_0 = c\check{h}$ in L^3, a critical space with respect to scaling. In this situation, we are able to adapt the local regularity theory of mild solutions of (NSE), as done by Kato in [10], to (gNSE).

The second issue in the construction of cascade solutions is the *existence of expectation*: it may happen that $\mathbb{E}_{\xi,t}|\mathbf{X}| = \infty$. Le Jan and Sznitman overcame this obstacle by introducing a simple condition $|\chi_0| \le 1$, which guarantees $|\mathbf{X}| \le 1$. The role of this condition is illustrated more clearly when h is the self-similar kernel. In this case, (FNS) has a scaling property $\chi_0(\xi) \to \chi_0(\lambda^{-1}\xi)$ and $\chi(\xi,t) \to \chi(\lambda^{-1}\xi, \lambda^2 t)$. In the terminology of Caffarelli-Kohn-Nirenberg, χ_0 and χ are dimensionless quantities. Then L^∞ is a critical space for both χ_0 and χ. Therefore, the result obtained by Le Jan–Sznitman is consistent with the rule of thumb that if the initial condition is sufficiently small in a critical space then a global strong solution exists.

On the other hand, it is known from the theory of mild solutions of (NSE) that smallness of u_0 in $\dot{H}^{d/2-1}$ implies a global mild solution. Its norm can be expressed in terms of χ_0 as

$$\|u_0\|_{\dot{H}^{d/2-1}} = C_d \left\{ \int_{\mathbb{R}^d} |\xi|^{d-2} h^2(\xi) |\chi_0(\xi)|^2 d\xi \right\}^{1/2}.$$

The smallness of u_0 in $\dot{H}^{d/2-1}$ amounts to smallness of χ_0 in a global (integral) sense. Such a condition allows $|\chi_0|$ to be large in certain regions. However, it is not clear from the cascade picture whether $\mathbb{E}_{\xi,t}|\mathbf{X}|$ is finite because branches could terminate in regions where $|\chi_0|$ is large. In my paper [22] with Enrique Thomann, we address the following questions:

> (Q) Can the smallness of χ_0 in some integral sense guarantee that the cascade solution is well-defined? What are some natural settings for χ_0 and χ other than L^∞? In what case can one achieve local (or global) existence and uniqueness of solutions? Do cascade solutions coincide with mild solutions of (FNS)?

Our main result can be stated roughly as follows:

Theorem 4 ([22]) *Let X be an adapted value space for (FNS), i.e. a setting for χ_0 where fixed point method works. Then cascade solution is well-defined almost everywhere up to the time when mild solution ceases to exist and coincides with the mild solution almost everywhere.*

In particular, the smallness of χ_0 in $\dot{H}^{d/2-1}$ indeed yields a global cascade solution χ. The key observation is that cascade solutions to (FNS) are dominated by cascade solutions of the so-called *majorizing Navier-Stokes equation.*

$$\text{(mNSE):} \quad \begin{cases} \partial_t u - \Delta u &= \sqrt{-\Delta}(u^2) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ u(\cdot,0) &= u_0 \quad\quad\quad\;\; \text{in} \quad \mathbb{R}^d. \end{cases}$$

The main difficulty is to find a suitable functional/norm to control the size of the cascade solution. Our introduction of admissible functionals enables us to do so. They are compatible with the norm of admissible path spaces in the fixed point method. In particular, when χ_0 belongs to an adapted value space, the cascade solution is the limit of a Picard iteration and, thus, coincides with mild solution. This observation is consistent with [1].

3 Minimal blowup data

The question whether there exists a *minimal* initial datum leading to a blowup solution of the 3D Navier-Stokes equations, under the assumption that there exists an initial datum leading to finite-time singularities, has attracted the interest of mathematicians in the recent years. On one hand, answers to this question can shed light on the behavior of blowup solutions if they exist. On the other hand, existence of minimal blowup data of a given differential equations is a topic of its own interest. For example, finding minimal blowup data can be seen as an optimization problem when the size of minimal blowup datum is interpreted as the minimal cost one must pay to generate a blowup solution. The question was addressed for the harmonic map heat flow in [2, 3] and semilinear Schrödinger equation in [24]. For the Navier-Stokes equations, the answer is affirmative in the settings that have been considered [6, 7, 8, 15, 23, 25]. In the following, I will describe two of my research projects on this topic. They will appear in [21] (an extraction of my PhD thesis [20]) and [4].

First, let us formulate the problem generally as follows. For $\Omega \subset \mathbb{R}^3$, consider the initial boundary value problem for the Navier-Stokes equations in $\Omega \times (0, \infty)$

$$(\mathrm{NSE})_\Omega : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases}$$

For $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$, the problem has a scaling symmetry $u_0(x) \to \lambda u_0(\lambda x)$ and $f(x,t) \to \lambda^3 f(\lambda x, \lambda^2 t)$. The initial condition u_0 and the external force f are assumed to be in critical spaces with respect to the natural scaling, namely X and Y respective. In general, it is known that $(\mathrm{NSE})_\Omega$ is globally well-posed in the class of mild solutions in $L^5_{t,x}$ for sufficiently small data $(u_0, f) \in X \times Y$. Global well-posedness for large data is still not known.

Let $\rho^\Omega_{\max}$ be the supremum of all $\rho > 0$ such that $(\mathrm{NSE})_\Omega$ is globally well-posed for every (u_0, f) with $\|(u_0, f)\|_{X\times Y} = \|u_0\|_X + \|f\|_Y < \rho$. For convenience, we will denote it as $\rho_{\max}$ if $\Omega = \mathbb{R}^3$, and $\rho^+_{\max}$ if $\Omega = \mathbb{R}^3_+$. Although it is not known whether $\rho^\Omega_{\max}$ is finite or infinite, we are interested in the hypothetical situation when $\rho^\Omega_{\max}$ is finite. We consider the question:

(Q) If $\rho^\Omega_{\max}$ is finite, does there exist a data $(u_0, f) \in X \times Y$ with $\|(u_0, f)\| = \rho^\Omega_{\max}$, such that the solution u of the system $(\mathrm{NSE})_\Omega$ blows up in finite time?

We call such a pair (u_0, f) a *minimal blowup data*. For $\Omega = \mathbb{R}^3$ and $f = 0$, affirmative answers are given in [6, 7, 8, 15, 25] for different choices of X, including L^3. It is known that physical boundaries complicate the regularity theory, especially the treatment of pressure at the boundary. In my upcoming paper with Vladimír Šverák [21], we study the influence of the boundary on the existence of minimal blowup data. The main difficulties are (1) the low regularity of pressure at the boundary, and (2) the instability of blowup solutions with respect to localization of domains. To deal with the second issue, we introduce the force term to make the definition of $\rho^\Omega_{\max}$ more stable under the change of domains and perturbation of equations. To deal with the the first issue, we introduced a class of weak solutions called *sw-solutions*, based on the idea of Seregin and Šverák [26]. This type of weak solutions is shown to be suitable for boundary regularity. We note that the following Lebesgue weighted critical spaces, as a choice for Y, is better suited for our analysis than the critical Lebesgue spaces $L^{5/3}$. Our method works well for both the half-space and the whole space.

$$Y_q = \{f : \Omega \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\Omega \times (0,\infty))\}$$

with $\|f\|_{Y_q} = \|t^{q_*} f\|_{L^q}$ and $q_* = 3/2 - 5/(2q)$. Our main result is the following.

Theorem 5 ([20]) *For $Y = Y_q$ with $5/2 < q < 3$, we have*

(a) If $\rho_{\max} < \infty$ then there exists a minimal blowup data for (NSE).

(b) $\rho^+_{\max} \le \rho_{\max}$.

(c) If $\rho^+_{\max} < \rho_{\max}$ then there exists a minimal blowup data for $(\mathrm{NSE})_+$.

When $\rho^+_{\max} < \rho_{\max}$, the boundary "facilitates" blowup in sense that all singularities, if exist, must stay within a finite distance from the boundary. The case $\rho^+_{\max} = \rho_{\max}$ happens only when the singularities move away from the boundary. In this scenario, the boundary seems to obstruct the existence of minimal blowup data.

While it is not known if a smooth, rapidly decaying initial datum could produce a finite-time blowup solution to (NSE), Montgomery-Smith [17] showed that it is the case for the majorizing

Navier-Stokes equation (mNSE) (which he called a *cheap Navier-Stokes equation*). This scalar equation has the same scaling symmetry as the Navier-Stokes equations. It is natural to ask whether minimal blowup data exist for (mNSE), for example, in the critical setting $u_0 \in L^3$.

The available methods for the Navier-Stokes equations rely on some types of local energy estimates. These estimates serve two purposes. First, they guarantee that the singularities are stable under weak limit of initial conditions. Secondly, they give global mild solutions decay as $t \to \infty$. In particular, if u belongs to L^5 for all finite times, it stays in $L^5(\mathbb{R}^3 \times (0,\infty))$. Unfortunately, there are no energy estimates that could serve similar purposes in (mNSE). In fact, the counter-example constructed by Montgomery-Smith has infinite kinematic energy at blowup time. In my upcoming paper [4] with Radu Dascaliuc, we study the existence of minimal blowup data for (mNSE) by Le Jan–Sznitman cascade method. We obtain the following.

Theorem 6 ([4]) *Consider initial condition of the form*

$$u_0(x) = \frac{\gamma}{1+|x|^2} \in L^3(\mathbb{R}^3). \tag{1}$$

For $\gamma \geq 2$, the mild solution to (mNSE) blows up at some $T \leq \infty$ in sense that $\|u\|_{L^5(\mathbb{R}^3\times(0,T))} = \infty$. For $0 \leq \gamma < 2$, the mild solution belongs to $L^5(\mathbb{R}^3 \times (0,\infty))$.

The fact that the Fourier transform of (mNSE) has the same cascade structure as (FNS) enables us to formulate the problem in Fourier domain. Specifically, for $\chi = c\hat{u}/h$ and $\chi_0 = c\hat{u}_0/h$,

$$\chi(\xi,t) = \sum_{k=1}^{\infty} \chi_0^k \, p_k(\xi,t)$$

where $p_k(\xi,t)$ denotes the probability that the binary tree rooted at ξ has exactly k branches crossing the horizon. If u_0 satisfies (1) and h is equal to the Bessel kernel, then χ_0 is a constant. The key observation is that the critical value $\gamma = 2$ corresponds to a time-independent solution u, which blows up at $T = \infty$ according to our definition. The main difficulty is how to obtain decay of χ as $t \to \infty$ in the case $0 \leq \gamma < 2$. We obtain an exponential decay through suitable estimates for p_k.

4 Other research plans

Pertaining to the project in Section 1, I plan to study the relation of ε (resolution parameter) and M (size of approximate solution) in the case of stationary NSE. This is possibly the setting where the ideal relation $\varepsilon \sim M^{-1}$ would hold. On the project in Section 2, I plan to investigate further the question: can stochastic explosion, which is now known for 3D self-similar cascade, provide a pathway to nonuniqueness of NSE in the class of cascade solutions?

As a young scholar, I enjoy learning different aspects of mathematical analysis, and I am open to participating in various research directions. Beside the Navier-Stokes equations, I am also interested in other PDE and stochastic PDE arising from turbulent flows, quantum physics, biology and other disciplines. I plan to explore one or more of these fields in the future.

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